CONTROL OF RIGID SATELLITES FOR LARGE ANGLE EIGEN-AXIS ROTATION USING QUATERNION FEEDBACK

by

MASHUQ-UN-NABI

DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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Certificate

It is certified that the work contained in the thesis entitled CONTROL OF RIGID SATELLITES FOR LARGE ANGLE EIGEN-AXIS ROTATION USING QUATERNION FEEDBACK by Mashuq-Un-Nabi, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

(Dr. V. R. Sule)
Associate Professor
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

April, 1999.

This piece of effort is dedicated

To

My Parents

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Abstract

Efficient control laws that ensure fast and accurate maneuver of Spacecrafts along with accurate and reliable control of the attitude are crucial for Space applications. In addition, the angular path traversed in such a maneuver is also an important aspect as this has a direct effect on the time as well as energy required during such maneuvers. In this thesis, the structure of Control laws necessary for a shortest-path Eigen-axis rotation between two attitudes are investigated. The attitude feedback taken is in terms of quaternion parameters which is a represention of attitude in 3-Dimentional space well suited for space-applications. Based on this, few Control laws are suggested. Global asymptotic stablity of the resulting close-loop systems are proved through Lyapunov stability analysis. Computer simulations of the proposed Control laws are carried out to depict the responses of the systems under various initial conditions.

Contents

Chapter 1

Introduction

In space-technology applications, the attitude of a spacecraft in orbit is often very important. For example, communication satellites may have to be pointed in a particular direction for recieving and beaming signals efficiently. Imaging satellites may be required to point accurately at some area on earth. For applications like space-telescopes, accurate steady pointing towards an object in space, for long periods of time, is absolutely crucial for observation and long-exposure photography. In applications like space-based defence, fast and accurate pointing and tracking of targets become extremely critical. In addition, in recent years, there are also increasing applications of giant robot arms in space for repairing, re-orienting, and retreiving satellites.

Evidently, it becomes extremely important to have a good control over the attitude of the spacecraft. In addition, other desired criteria like low fuel consumption, fast response, etc. make the attitude control of spacecrafts in orbit an important and challenging area of control applications.

Attitude control of spacecrafts may be broadly categorised into two types. In some

cases, the body is desired to maintain a particular orientation in space. The control problem thus consists of bringing the body from any arbitrary initial orientation to this particular orientation. This class of control systems are termed as *regulators*, or of the *regulatory* type. On the other hand, in many applications the spacecraft is desired to move in a particular manner tracing a pre-defined trajectory. Here the control problem involves taking the body along this trajectory. This class of control systems are termed as those of the *tracking* type.

In addition, another area in which the orientation of solid body in 3-Dimentional space is sought to be controlled, is robotics. In robotics, robotic hands are used for working tools, as in assembly lines, in handling materials, and similar cases. In these areas, often the fingers of the robot-arm act like actuators in controlling attitude of another rigid body, like a tool or vessel. Due to the inherent similarity of the control problems involved, many techniqes, and developments in the control of spacecrafts can be applied to robotics.

1.1 Representation of attitude and its control

For the purpose of analysis of general rigid-body motion, two reference-frames are considered. each consists of a set of three right-handed cartesian axes. One of them is fixed on the rigid-body, and is termed the *body-fixed frame*. The other is fixed in inertial space, and is called the *inertial reference frame*. For motions in which there is no translation, a point on the body always remain fixed in inertial space. This fixed point can be taken as the common origin of both the frames.

With this, the attitude of the body may be defined as the relative orientation of the two reference frames. We define the *The reference attitude* of the body as that attitude for which these two frames coincide. Other kinematic quantities like angular

velocity, acceleration etc. are similarly the relative velocities etc of the bodyfixed frame relative to the inertial frame.

1.1.1 representation of attitude

The attitude of the body can be represented mathematically by any set of parameters describing the relative orientation between these two frames. Based on this, there are a few conventions of attitude representation. Most important among these are mentioned as follows.

1. **Euler angles** : This is one of the most used representations of attitude. It represents the attitude in terms of three successive rotations of the body-fixed frame with respect to the inertial frame, where the rotations are about the body-fixed axes. The angles of these rotations, denoted by ϕ, θ and ψ, are called the 'Euler angles'.

2. **Quaternions** :This method of attitude parametrization is based on 'Euler's theorem for rigid-bodies'. It represents the attitude in terms of a quadruple of real numbers, or to be more precise, a scalar and a 3-Dim vector. Due to its suitability in numerical processing and some other desirable properties, it has come to be widely used in recent years, especially in spacecraft applications.

3. **Cayley-Rodrigues parameters** : This is another 4-parameter represention of 3-Dim attitude. Closely related to quaternions and Cayley-Klein parameters which also share the fast-numerical analysis advantage like the quaternions. They are applied often in other branches like physics wher rotaional symmetry is present.

1.1.2 Mechanisms for attitude control

In space applications, various types of mechanisms are used to produce the control torques required for executing maneuvers. Two major categories of these are

- Reaction wheels
- Retro rockets

In reaction-wheel mechanisms, the spacecraft has mechanical wheels along the three axes. These are called reaction-wheels or momentum-transfer-wheels. By transfering the spacecraft's momentum to these wheels, thereby redistributing the total momentum along the three axes, a desired attitude of the body can be achieved [3].

In the retro-rocket method, small rockets or jets on the spacecraft are fired to deliver directional thrusts to reorient the body. these jets are mostly either pulse-width-modulated (PWM), pulse-frequency-modulated (PFM), or a combination of both (PWPF) [6].

1.2 Previous work

The control of attitude and motion of a spacecraft has been an area of considerable investigation for quite a long time. A fair amount of published literature can be found, which describes the various approaches regarding attitude representation, and mechanisms used.

In the area of attitude representation [4] gives two numerically efficient algorithms for determination of spacecraft attitude from vector observations. A basic introduction to the use of quaternions for spacecraft attitude representation is given in [2]. An algorithm for global attitude estimation in quaternions can be found in [5].

The specific problem of stopping a rotating spacecraft while optimizing an energy function, has been described in [13]. In [6], it was shown that for attitude-

feedback in terms of quaternions, a large angle stable rest to rest maneuver can be achieved. A fairly detailed study of the attitude control problem and various control laws using quaternions is done in [11]. In [10] a linearization of the attitude error dynamics is shown, while [9] gives a control law in which the angular rates need not be actually measured, thus requiring no velocity sensors. A globally assymptodically stable control law that takes the spacecraft from any arbitrary orientation to any other orientation using quaternion feedback, was suggested in [8]. But there, the gains selected were arbitrary, with no optimization procedure suggested. In [7], however, a control law achieving a large-angle maneuver through the shortest-angular-path Eigen-axis rotation was suggested. This is optimal in path-length, but no method or possibility of optimization among such laws with respect to energy or time was suggested.

1.3 Preview of problem considered in this thesis

In this thesis, we derive control laws for taking a spacecraft from any arbitrary initial attitude to any other desired attitude, through an Eigen-axis rotation. The form of control law derived is a generalization of the control law suggested in [7]. The form suggests a family of possible control laws for achieving the above objective. Possible scopes of optimization over this family are also suggested.

1.4 Organization of this thesis

In Chapter 2, a brief introduction to quaternions and their use in attitude representation and rotational kinematics is given. The Kinematic and Dynamic equations governing rotating rigid-bodies are presented in terms of quaternion parameters. The concept of Eigen-axis rotation is introduced and its mathematical representation is derived.

In Chapter 3, a detailed study into the structure of Control laws required for Eigen-axis rotation is done. This leads to suggestion of a number of control laws for the above purpose. Stability analysis of those Control laws is done, followed by a brief robustness analysis.

Chapter 4 presents the results of Computer simulation of the Close-loop Control systems with the developed Control laws along with brief discussions on them.

In Chapter 5 suggestions regarding possible future work are given.

Chapter 2

Quaternion representaion of system equations

In this chapter we present the kinematic and dynamic equations of a rigid-body moving with one point fixed, in terms of quaternions. Traditionally, rigid-body kinematics is handled through the Euler-angle formulation of attitude. Hence, the kinematic and dynamic equations have been given in terms of the Euler angles ϕ, θ and ψ along with ω_i, the angular velocity components along any of the two frames, body-fixed or inertial [1]. However these equations involve trigonometric functions of ϕ, θ and ψ which hinder fast and efficient numerical processing. In addition, they also involve singularities in attitude representation. Nevertheless, despite these drawbacks, Euler-angles have remained one of the most popular formulations.

The kinematic equations in terms of quaternions as attitude parameters was first given in 1958. For a short history of quaternionic attitude representation and kinematic equations, the reader is referred to [7] and [2]. In recent years, quaternions have been commonly used as attitude parameters mainly in space applications due to some of their important and useful properties. These can be mentioned as follows.

1. They have no inherent goemetrial singularities, as do Euler-angles.[6] [8] [9] [11]

2. They have no singularities in their kinematic differential equations, as do the Cayley-Rodrigues parameters. [6]

3. Successive rotations can be represented as successive multiplications by 4×4 quaternion matrices.[2] [6] [9]

4. They are algebraic quantities, with only products appearing in all their equations, without any trigonometric quantities. This makes them suitable for fast numerical processing on board spacecrafts.[2] [6] [8] [11]

The above factors make them ideal for use in space applications.

2.1 Euler's theorem and Euler-rotations

The quaternion approach to representation of attitude and rotations is based on the well Euler's theorem for rigid body motion. The theorem can be stated as follows

Theorem 1 (Euler's theorem for rigid-bodies) *' The most general displacement of a rigid body , with one point fixed , is a rotation about some axis .'*

Let a rigid body execute a single, or a sequence of maneuvers, such that one point on the body always remain fixed in an inertial frame. Then the above theorem implies that the net resultant transformation from the initial orientation to the final, is always equivalent to one single rotation about an axis passing through the fixed point. The theorem also implies that for a rigid body moving with one point fixed, the resultant attitude after any number of transformations starting from the reference attitude, can always be expressed as the result of a single rotational maneuver from the reference to the final present attitude. The axis of this rotation is called the *Eigen-axis* or *Euler-axis* for that attitude. The point that remains fixed obviously lies on this axis. A

rotation about the Eigen-axis, is called an *Eigen-axis rotation* or *Euler rotation.* The body then be brought back to the initial or reference orientation by the same path in reverse. That is, rotating the body about the Eigen-axis would enable it to return to the original orientation through a single rotational maneuver, irrespective of the type and number of maneuvers that took the body to the final orientation. The physical facts mentioned above can also be understood from a mathematical view-point. For this, and a detailed discussion of the theorem, the reader is reffered to the Appendix. It can be shown that for a given initial or reference attitude and a final attitude, the Eigen-axis is unique, irrespective of the intermediate maneuvers. (Appendix) Hence, for a given reference attitude, any other attitude can be uniquely specified in terms of its Eigen-axis and the angle of the Eigen-axis rotation. Thus these two can serve as parametrization of attitude for a rigid-body moving with one point fixed.

The case is equivalent to the linear case of a point-object moving in 3-Dim space starting from a location and reaching a final location after a series of moves. In Fig 2.1 Let P_0 be the initial position, and a reference frame $(X - Y - Z)$ fixed with P_0 as the origin. Now the displacement of the body from initial P_0 to final P_n can be aways be considered equivalent to three consecutive independant displacements parallel to the three axes, as shown. In particular they can be reversed to return the body to P_0 irrespective of how it has actually come to the point P_n. Hence the lengths of these three moves are properties of the location, and can be used to spaecify it — being called the *co-ordinates* of the location.

2.2 Euler's theorem and quaternions

As seen in the previous section, the attitude of a rigid body can be specified in terms of an axis, say $(n_1 \, n_2 \, n_3)$, and a corresponding rotation about it through an angle, say $'\theta'$, that would take the body from the reference or origin attitutude to the present

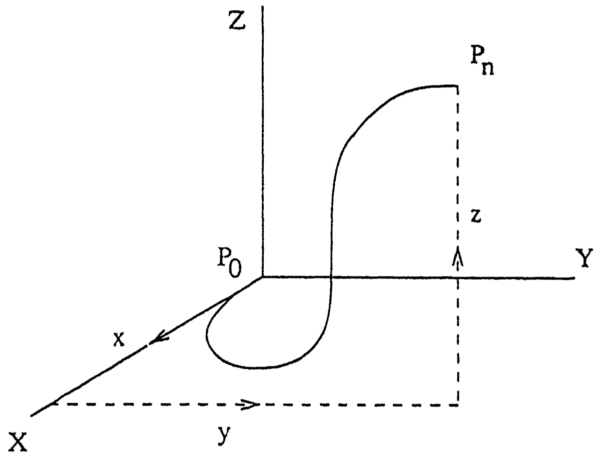

Figure 2.1: The co-ordinates of a point

one.

Based on this, a quadruplet of real numbers given by the following can be used as an attitude or orientation measure.

$$q = (q_0 \, , \, \bar{q}\,) = (q_0 \, , \, q_1, q_2, q_3\,)$$

where

$$q_0 \;=\; \cos\theta/2 \tag{2.1}$$

$$\bar{q} \;=\; \hat{n}\cdot\sin\theta/2 \tag{2.2}$$

and

$$\hat{n} = (n_1, n_2, n_3)$$

is a unit vector along the axis of rotation (Eigen-axis). Here $\bar{q}$, which is clearly collinear with $\hat{n}$ is called the vector part of the quaternion , and q_0 the scalar part.

Equations 2.1 and 2.2 can be expanded into the form

$$\begin{aligned} q_0 &= \cos\theta/2 \\ q_1 &= n_1 \sin\theta/2 \\ q_2 &= n_2 \sin\theta/2 \\ q_3 &= n_3 \sin\theta/2 \end{aligned}$$

It can be checked that

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Thus in the representation of attitude, the quaternions are of magnitude one, that is, they are *unit quaternions* . It is easy to see that the original or reference orientation of the body would correspond to the case $\theta = 0$. with $\hat{n}$ being any vector whatsoever. Physically, it means the body has not undergone any rotation about any axis whatsoever. For this attitude, irrespective of $\hat{n}$, the quaternion representation is always

$$q_e = (q_0 \,,\, \bar{q})_e = (1\,,0\,,0\,,0)$$

Various ways of attitude detremination in terms of quaternions, using an inertial reference frame,can be found in [4], [2], and [12]. In short, the relation between the quaternion representation and the Direction-cosine and Euler-angle representations, can be understood through the following observations. The orthogonal transformation matrix or the rotation matrix, corresponding to a rotation about the axis $\hat{n}$ through an angle θ is given by

$$A = 2\begin{bmatrix} q_1^2 + q_0^2 - \frac{1}{2} & q_1 q_2 - q_3 q_0 & q_1 q_3 + q_2 q_0 \\ q_1 q_2 + q_3 q_0 & q_2^2 + q_0^2 - \frac{1}{2} & q_2 q_3 - q_1 q_0 \\ q_3 q_1 - q_2 q_0 & q_3 q_2 + q_1 q_0 & q_3^2 + q_0^2 - \frac{1}{2} \end{bmatrix} \tag{2.3}$$

where q_0, q_1, q_2, q_3 are related to θ and $\hat{n}$ by the Eqs 2.1 and 2.2. The above can be derived from a basic geometric consideration of the rotation (Appendix) and is related to the well-known 'rotation-formula' and the 'Rodrigues formula ' in mechanics. Conversely, if the rotation matrix A is given, say as the Direction-cosine matrix itself or in terms of Euler-angles , then the vector $\hat{n}$ is easily recognised as the Eigen-vector corresponding to the Eigen-value 1 of the matrix and the angle θ is found by noting that the other two Eigen-values are $e^{i\theta}$ and $e^{-i\theta}$. (Appendix). Thus, by considering Eqs 2.1 and 2.2, q may be found .

A much more direct way of calculating q, form the rotation matrix A is sugested by Eq 2.3. Looking at Eq 2.3, we can observe that

$$1 + trace[A] = 4\,q_0^2$$

By using the constraint

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

we have

$$q_0 = \frac{1}{2}\sqrt{1 + trace[A]} \tag{2.4}$$

Further, it can be seen that

$$A - A^T = 4\,q_0\,Q_\times$$

where $Q_\times$ is the cross-product matrix of $\bar{q}$, given by

$$Q_\times \equiv \bar{q}\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

Thus, from the above, and using Eq 2.4, we have

$$Q_\times = \frac{1}{2} \cdot \frac{1}{\sqrt{1 + trace[A]}} \cdot (A - A^T) \tag{2.5}$$

Hence, both q_0 and $\bar{q}$ are found. Generally, both $\bar{q}$ and $-\bar{q}$ may represent A as mentioned in [9], but the kinematic equations considered here (Eq 2.7), as in [9], are consistent with $\bar{q}$ representing A. A numerically efficient algorithm for computing q from the rotation matrix can be found in [12].

The above discussion shows that the numerical and control processings can be done in terms of quaternions, even if the attitude is measured actually in terms of other parameters like Euler-angles. In these cases, the above relations, or the algorithms in [9] may be used to transform the attitude representations to quaternions.

2.3 The Control Problem

In most control applcations, the control laws are designed to take the body to an orientation for which the quaternion representation is $(1,0,0,0)$. As seen in the previous section, this is a representation of the reference attitude of the body. When this is the target value of the quaternion $q = (q_0, \bar{q})$, the value of $\bar{q}$ at an instant is itself an error measure. Hence, the quaternion $q_e = (1,0,0,0,)$ is taken as the 'origin' or 'zero' attitude, and the control problem formulated as a regulator problem. Control laws are developed which take the spacecraft from a non-zero orientation , represented by

$$q(0) = (q_0(0)\,,\,\bar{q}(0)) \quad ; \quad \bar{q}(0) \neq 0$$

to the origin or 'zero' orientation

$$q_e = (1\,,0\,,0\,,0)$$

However this does not restrict the generality of the attainable target attitude. In other words, this does not imply that the control laws will take the spacecraft always to a fixed particular orientation in 3-Dimentional space. This is because *any* targetted orientation of the body can be taken as the reference attitude corresponding to the

origin q_e, and the present orientation $q(0)$ is expressed with reference to this, thus having non-zero $\bar{q}(0)$. Hence the problem taking the body from any given present orientation to any other target orientation, is translated to one in which it is taken from an initial attitude $q(0)$ with non-zero $\bar{q}(0)$ to $(1,0,0,0)$.

This formulation is an often-used one and has been used in [6],[7] and [8]. A detailed discussion regarding possible other attitude error measures using quaternions can be found in [11] .

2.4 Kinematic and Dynamical Equations

In terms of conventional Euler angles ϕ , θ and ψ, the kinematic equations can be written as follows.

$$\begin{aligned}
\dot{\phi} &= (\sin\psi/\sin\theta)\,\omega_1 + (\cos\psi/\sin\theta)\,\omega_2 \\
\dot{\theta} &= (\cos\psi)\,\omega_1 - (\sin\psi)\,\omega_2 \\
\dot{\psi} &= -(\sin\psi\cot\theta)\,\omega_1 - (\cos\psi\cot\theta)\,\omega_2 + \omega_3
\end{aligned}$$

Here ω_1 , ω_2 and ω_3 are the angular velocities about the three body-axes. It should be noted that similar equations can be written where ω_i are components of the angular velocity in inertial frame. Also, various conventions of taking the Euler-angles give rise to various similar but different equations [1]. The dynamic equations relating the angular-velocities and the external torques are given by the well-known *Euler equations* of rigid-body motion. For a general case when the body-fixed axes may not coincide with the principal axes, they are written as

$$J\dot{\omega} = -\omega \times J\omega + \tau \tag{2.6}$$

where $J_{3\times3}$ is the inertia matrix, ω the angular velocity vector and τ the torque vector. A detailed discussion and derivation of the above equations can be found in

any standard treatise on rigid-body mechanics .

Next, we consider the kinematic equations of rigid-body motion with one point fixed, where the attitude parameters are quaternions. The are stated as follows

$$\dot{q} = \frac{1}{2} F(q) \cdot \bar{\omega} \tag{2.7}$$

where

$$\begin{aligned} q &= [\ q_0\ \ q_1\ \ q_2\ \ q_3\ ]^T \\ \bar{\omega} &= [\ \omega_1\ \ \omega_2\ \ \omega_3\ ]^T \end{aligned}$$

and $F(q)$ is the matrix given by

$$F(q) = \begin{bmatrix} -\bar{q}^T \\ q_0 I_{3\times3} - Q_\times \end{bmatrix} = \begin{bmatrix} -q1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix}$$

It may be observed that the quaternion-kinematic equation 2.7 involves only algebraic terms, while those for Euler-angles involve trigonometric terms, illustrating the numerical advantage of quaternions.

Further, the matrix $F(q)$ can be seen to satisfy the following important relations

$$F^T(q) \cdot F(q) = I_{3\times3} \quad ; \quad F^T(q) \cdot \bar{q} = 0$$

Using the above, and Eq 2.7, we can obtain

$$q^T \dot{q} = \frac{1}{2} q^T \cdot F(q) \cdot \bar{\omega} = 0 \cdot \bar{\omega} = 0$$

which implies

$$q^T q = Constant$$

This is consistent with the constraint on the norm of q, always being 1. In other words, the solutions of Eq 2.7 are all unit-quaternions, representing attitude.
Eq 2.7 can be written separately for the scalar and vector parts, as

$$\dot{q}_0 = -\frac{1}{2}\bar{q}^T\bar{\omega} \tag{2.8}$$

$$\dot{\bar{q}} = \frac{1}{2}q_0\,\bar{\omega} - Q_\times\,\bar{\omega} \tag{2.9}$$

$$= \frac{1}{2}q_0\,\bar{\omega} - \bar{q}\times\bar{\omega} \tag{2.10}$$

The dynamical equations for rigid rotating bodies are the same as in Euler-angle formulations, that is same as Eq 2.6. Hence, the Eqs 2.8, 2.10 and 2.6 are the equations describing the spacecraft motion. Then, considering the torque vector τ as the input vector the control system equations in terms of quaternions are

$$\dot{q}_0 = -\frac{1}{2}\bar{q}^T\,\bar{\omega} \tag{2.11}$$

$$\dot{\bar{q}} = \frac{1}{2}\Omega\bar{q} + \frac{1}{2}q_0\,\bar{\omega} \tag{2.12}$$

$$J\dot{\bar{\omega}} = -\Omega\,J\,\bar{\omega} + u \tag{2.13}$$

where Ω is the cross-product matrix of $\bar{\omega}$, given by

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega 1 & 0 \end{bmatrix}$$

The equations no 2.11, 2.12,and 2.13 give our system state-space equations , with

$$[\ q_0\ q_1\ q_2\ q_3\ \omega_1\ \omega_2\ \omega_3\ ]^T$$

as the states and

$$[\ u_1\ u_2\ u_3\ ]^T$$

as the controls.

2.5 Euler-rotation and its mathematical expression

We expalained above how a manuever of a rigid-body with one point fixed from one attitude to any other can be achieved by a single rotation of the body about an axis. The axis is called the Eigen-axis or Euler axis, and given such an axis $\hat{n}$, a rotation of the body about this axis is called an Eigen-axis rotation or Euler-rotation. Morever, we can take the target attitude as the reference attitude with the quaternion representation $(1, 0, 0, 0)$. The above formulation is used for developing control laws for changing the attitude of the attitude while the initial attitude is represented by an arbitrary quaternion

$$q(0) = (\ q_0(0)\ ,\ \bar{q}(0)$$

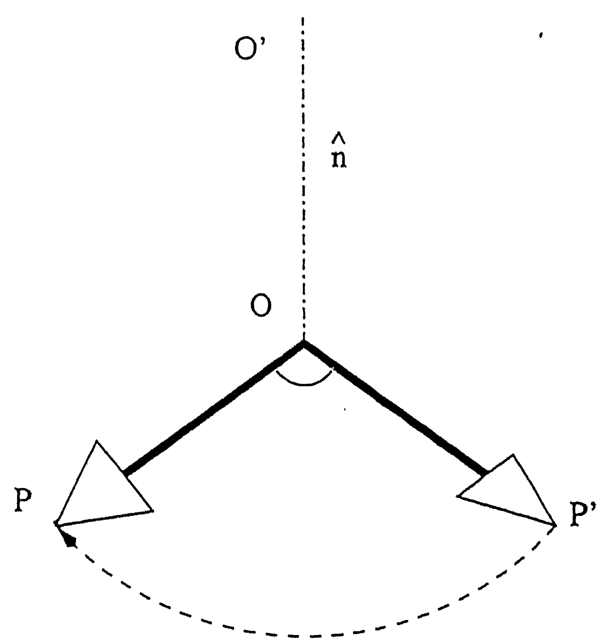

Figure 2.2: Eigen-axis rotation

Referring to the Fig 2.2, let the initial attitude of the body be depicted by a certain position vector OP' of a point P' while the target attitude has the same

position vector rotated to OP through an angle θ_0. OO' is the Eigen-axis, with unit vector $\bar{n}$ and the rotation trajectory is as shown. Recall that the vector $q(0)$ is related to $\bar{n}$ and the angle θ of the Eigen-axis rotation through the Eq 2.1 and 2.2.

As the body traces the shown Eigen-axis trajectory to the target orientation OP , it is clear from the figure that all intermediate orientations also have the same Eigen-axis $\hat{n}$, while they have continiously decreasing θ from θ_0 to 0. Hence , The attitude of the body undergoing Eigen-axis rotation will be such that the instanteneous vector part of the quaternion is given by

$$\bar{q}(t) \quad = \quad \hat{n} \cdot \sin\theta(t)/2$$

which always remains along $\hat{n}$. The direction of the angular velocity vector, will also evidently remain along the direction of $\hat{n}$ or $\bar{q}$ throughout, the magnitude starting from zero and returning to zero at the target point .

Hence , we can write , for the body to execute Eigen-axis rotation, the solution to the Kinematic and Dynamic equations 2.11,2.12 and 2.13should have the form

$$\begin{aligned} \bar{q} &= C_q(t) \cdot \bar{q}(0) \\ \bar{\omega} &= C_\omega\,(t) \cdot \bar{q}(0) \end{aligned}$$

with

$$\begin{aligned} C_q(0) = 1 \quad &; \quad C_q(t_f) = 0 \\ C_\omega\,(0) = 0 \quad &; \quad C_\omega\,(t_f) = 0 \end{aligned}$$

where t_f is the final time.

The objective of the control law for an Eigen-axis rotation is to achieve the above solutions of the control system equations 2.11,2.12 and 2.13 once the control law has been substituted into the eq 2.13. A detailed analysis of such control laws is carried out in the next chapter.

Chapter 3

Closed-loop Control laws for Eigen-axis Rotation

In this chapter we study the structure of a special form of control law that takes a spacecraft from any initial orientation to any given target orientation, through an Eigen-axis rotation as described in the previous chapter. A broad outline of the formulation of the problem was also given in that chapter. Based on this, we develop few control laws that are found to be generalizations of a control law given in [7] for the same objective. Global asymptotic stability for these control laws are shown, and where required, the conditions on these control laws needed to guarantee global asymptotic stability are also determined.

3.1 Developing the Control law

In [7], it was shown that a large angle rest-to-rest manuever about the Eigen-axis can be achieved by a quaternion feedback control law, with proper selection of feedback gain matrices of the quaternion feedback regulator. The control law suggested fir this

was of the form

$$u = \Omega J\omega - D\bar{\omega} - K\bar{q}$$

where the gain matrices K and D were given by

$$K = kJ \quad ; \quad D = dJ$$

k and d being two scalar constants. In that work, however no analysis leading to the suggested form was given. Further, it appeared that the restriction of the possible gain matrices to constant multiples of the inertia matrix was rather severe. In this chapter we try to arrive at a more general form of the control law required to achieve an Eigen-axis rotation, starting the analysis from the required form of the Eigen-axis trajectory. The more general form that we try to develop consists of general time-dependent gains.

3.1.1 The Control Problem

The quaternion kinematical equations are

$$\begin{aligned} \dot{q}_0 &= -(1/2)\,\bar{q}^T\bar{\omega} \\ \dot{\bar{q}} &= (1/2)\,\Omega\bar{q} + (1/2)\,q_0\,\bar{\omega} \end{aligned}$$

where $\quad q_0^2 + \bar{q}^T\bar{q} = 1 \quad ;$

and Ω represents the cross-product matrix of the angular velocity vector ω

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \equiv \bar{\omega}\times$$

The dynamical Euler's equation are given by

$$J\dot{\bar{\omega}} = -\Omega J\omega + u$$

where J is the 3×3 Inertia matrix and u the external torques. With this the control-system equations are

$$\dot{q}_0 = -(1/2)\, \bar{q}^T \bar{\omega} \tag{3.1}$$

$$\dot{\bar{q}} = (1/2)\, \Omega \bar{q} + (1/2)\, q_0 \bar{\omega} \tag{3.2}$$

$$J\dot{\bar{\omega}} = -\Omega J \omega + u \tag{3.3}$$

with

$$(q_0, \bar{q}, \bar{\omega})^T = (q_0, q_1, q_2, q_3, \omega_1, \omega_2, \omega_3)^T$$

as the state vector, and

$$u = (u_1, u_2, u_3)^T$$

as the input vector.

The problem considered is to find input u so that the solution to the above system is of the form

$$\bar{q} = C_q(t) \cdot \bar{q}(0) \tag{3.4}$$

$$\bar{\omega} = C_\omega(t) \cdot \bar{q}(0) \tag{3.5}$$

where $\bar{q}(0)$ is the initial value of the vector $\bar{q}$ and $C_q(t)$ and $C_\omega(t)$ are scalar functions of time, with

$$C_q(0) = 1 \quad ; \quad C_q(t_f) = 0$$

$$C_\omega(0) = 0 \quad ; \quad C_\omega(t_f) = 0$$

where t_f is the final time.

3.1.2 A special form of the control law

We choose a general feedback law involving the quaternion vector and the angular velocity, given by

$$u = \Omega . J\bar{\omega} - M\bar{\omega} - N\bar{q} \tag{3.6}$$

Where M and N are 3×3 time-varying matrices . We need to find suitable M and N so that with the above input the solutions to the close-loop system equations describe an Eigen-axis rotation. Substituting Eq 3.6 into Eq 3.3, it gives

$$\dot{\bar{\omega}} = -J^{-1}M\bar{\omega} - J^{-1}N\bar{q}$$

or

$$\dot{\bar{\omega}} = -A\bar{\omega} - B\bar{q} \tag{3.7}$$

where

$$A = J^{-1}M \quad ; \quad B = J^{-1}N$$

Hence, Eq 3.1, 3.2 and 3.7 are the state equations describing the close-loop control system which should have solutions of the form

$$\begin{aligned} \bar{q} &= C_q(t) \cdot \bar{q}(0) \\ \bar{\omega} &= C_\omega(t) \cdot \bar{q}(0) \end{aligned}$$

Hence, substituting these in Eqs 3.2 and 3.7, the necessary conditions for Eigen-Axis Rotation become the existence C_ω and C_q such that

$$\begin{aligned} \dot{C}_q\,\bar{q}(0) &= \tfrac{1}{2}C_qC_\omega\,\bar{q}(0) \times \bar{q}(0) + \tfrac{1}{2}q_0C_\omega\bar{q}(0) \\ \Rightarrow \qquad \dot{C}_q &= \tfrac{1}{2}q_0C_\omega \\ &= \tfrac{1}{2}C_\omega\sqrt{1 - Q^2C_q^2} \end{aligned}$$

with

$$Q = \| \bar{q}(0) \|^2 \quad .$$

and

$$\begin{aligned} \dot{C}_\omega \bar{q}(0) &= -AC_\omega \bar{q}(0) - BC_q \bar{q}(0) \\ \Rightarrow \qquad \dot{C_\omega} &= -AC_\omega - BC_q \end{aligned}$$

Thus in order to have an Eigen-axis rotation , the following equations should be satisfied

$$\dot{C}_q = \frac{1}{2} q_0 C_\omega = \frac{1}{2} C_\omega \sqrt{1 - Q^2 C_q^2} \tag{3.8}$$

$$\dot{C}_\omega I = -AC_\omega - BC_q \tag{3.9}$$

where I is the 3×3 identity matrix. We need to ascertain the conditions on A and B so that a scalar solution C_ω to Eq 3.9 exists .

3.1.3 Structure of matrices A and B

To investigate the structure of matrices A and B, we breakup Eq 3.9 into two parts, one involving the diagonal elements and the other involving the off-diagonal elements. We write them separately as

$$\dot{C_\omega} = -a_{ii}\, C_\omega \; - b_{ii}\, C_q \qquad \text{for } i = 1, 2, 3 \tag{3.10}$$

and

$$a_{ij}\, C_\omega \; + b_{ij}\, C_q = 0 \qquad \text{for } i = 1, 2, 3\ , i \neq j \tag{3.11}$$

We analyse the two separately as below.

The diagonal elements :

The three equations

$$\dot{C_\omega} = -a_{ii}\, C_\omega \; - b_{ii}\, C_q$$

will have a unique solution if and only if

$$a_{11} C_\omega \; + b_{11} C_q = a_{11} C_\omega \; + b_{11} C_q = a_{11} C_\omega \; + b_{11} C_q = \lambda \tag{3.12}$$

where λ is an arbitrary variable differentiable with respect to time. Thus the above, is a condition which must be obeyed by the diagonal elements of matrices A and B for achieving Eigen-axis rotation. Now, for developing such control laws, we observe that Eq 3.12 can be satisfied by the diagonal elements in any of the two following cases.

(a)

$$\begin{aligned} a_{11} &= a_{22} = a_{33} = a_d \\ b_{11} &= b_{22} = b_{33} = b_d \end{aligned}$$

where a_d and b_d are two arbitrary scalar variables, differentiable with respect to time. For this, Eq 3.10 becomes

$$\dot{C_\omega} = -a_d\, C_\omega - b_d\, C_q \qquad (3.10\ \mathrm{b})$$

(b)

$$b_{ii} = \{\lambda - a_{ii}\}/C_q$$

where λ is any time-differentiable function. For this case, Eq3.10 becomes

$$\dot{C_\omega} = -\lambda \qquad (3.10\ \mathrm{c})$$

<u>**The off-diagonal elements :**</u>

The six equations in Eq 3.11

$$a_{ij}\, C_\omega + b_{ij}\, C_q = 0 \qquad \text{for } i \neq j; i,j = 1,2,3$$

can be satisfied only through the following two cases.

(a) $a_{ij} = b_{ij} = 0$ for $i \neq j$; $i,j = 1,2,3$

(b) $b_{ij} = -\frac{C_\omega}{C_q} \cdot a_{ij}\ i \neq j$; $i,j = 1,2,3$

Any of the above will ensure satisfaction of Eq 3.11, which is required for an Eigen-axis rotation. But it should be noted that Eq 3.11 implies that the contribution of the

off-diagonal elements to the right-hand expression of Eq 3.9 is zero. Thus in either of two cases above, the off diagonal elements a_{ij} and bij for $i \neq j$ do not in any way contribute to the dynamics of the system. Therefore it is reasonable to consider, for developing control laws, only the case (a) for which the off-diagonal elements are simply taken as zero.

Hence, finally combining the results of the analysis of the diagonal and off-diagonal parts as above, we get two sufficient sets of conditions on the elements of A and B for achieving Eigen-axis rotation. They are given below.

(1)

$$\begin{aligned} a_{11} &= a_{22} = a_{33} = a_d \\ b_{11} &= b_{22} = b_{33} = b_d \\ a_{ij} &= b_{ij} = 0 \end{aligned}$$

(2)

$$\begin{aligned} b_{ij} &= \{\lambda - a_{ij}C_\omega\}/C_q \\ a_{ij} &= b_{ij} \qquad = 0 \end{aligned}$$

For this thesis, we will consider only the simpler form of the control law arising from the conditions of set (1) above. The matrices A and B that follow from these conditions are

$$\begin{aligned} A &= a_d I = J^{-1}M \\ B &= b_d I = J^{-1}N \end{aligned}$$

The above gives a condition on M and N , so that the body undergoes an Eigen-axis rotation . Changing a notation , writing two generally time-varying scalar functions $\psi(\cdot)$ and $\phi(\cdot)$ in place of a_d and b_d , we finally arrive at

$$\begin{aligned} M &= \psi(\cdot)\ J \\ N &= \phi(\cdot)\ J \end{aligned}$$

Thus the result is proved and we can write the control laws that achieve Eigen-axis rotation as

$$u = \Omega J\bar{\omega} - \psi J\bar{\omega} - \phi J\bar{q} \tag{3.13}$$

A block diagram of the resulting Control System is shown in Fig 3.1

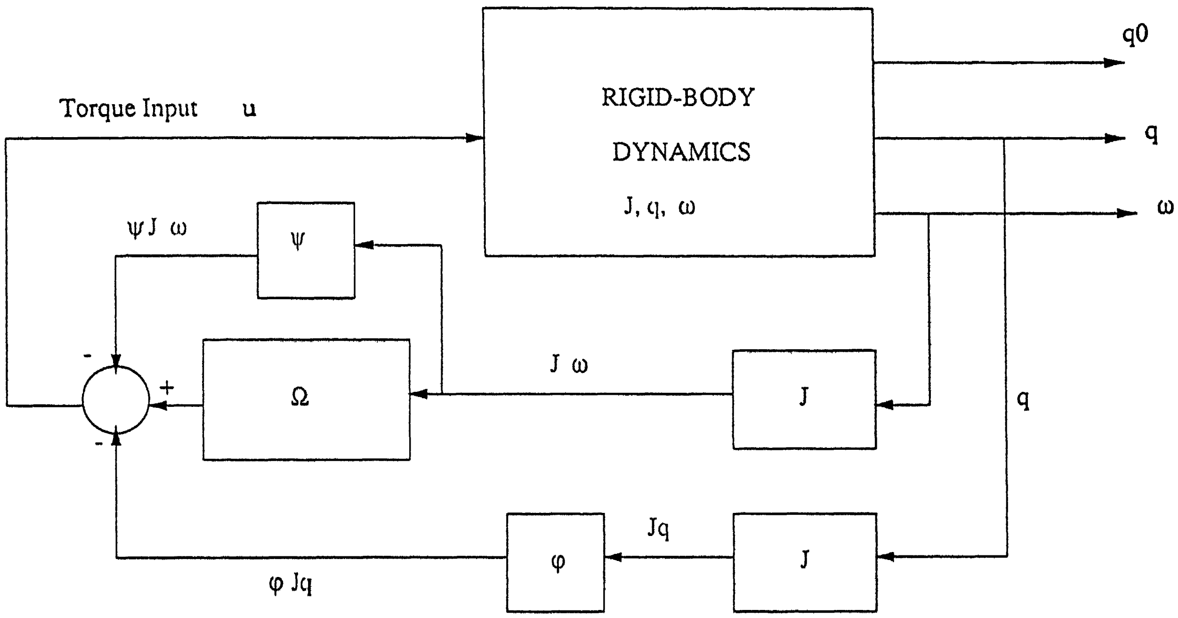

Figure 3.1: The Control System

3.1.4 Remarks

It may be mentioned that the first term was introduced to cancel the cross-coupling torque. In most cases this is found to be generally much smaller than the other torques involved [7]. As a result, even the exclusion of this term does not affect system response significantly. This fact was verified in our simulations of the above control laws. The reason behind the above is that in most cases, the principal moments of inertia of the spacecraft are considerably larger than the cross moments of inertia, as a result of their being generally symetrical . In addition, the body fixed axes are taken to coincide with the body's primary axes. In either case the inertia matrix J

is nearly or completely diagonal, making the vectors $\bar{\omega}$ and $J\bar{\omega}$ nearly or completely parallel. As a consequence the term $\Omega\ J\bar{\omega}$, which is $\omega \times J\bar{\omega}$,is small or zero.

Therefore, this term is required only in those cases where this cross-coupling torque is significantly large.

We thus conclude that the control law of Eq 3.6 need not be restricted to those with gain matrices being a constant multiple of the inertia matrix. We can have a family of control laws with different time-varying ϕ and ψ, They can be purely time-functions or may be also be taken as scaler functions of the state $(\bar{q}, \bar{\omega}\ )$. The control law suggested in [7] is evidently a special case fo this, with both ϕ and ψ as constants. This leaves us with a scope of optimization over a number of control laws, with respect to some objective function. This also requires a study of the conditions on ϕ and ψ, for the system to be stable.

3.2 Stability analysis

In this section, we determine conditions on time-varying functions $\phi(t)$ and $\psi(t)$ so that the close-loop system has the point $(q_0 = 1, \bar{q} = 0, \bar{\omega}\ = 0)$ as a stable equilibrium.For this we make use of Lyapunov's second or direct method for a general time-varying system. The methosd is based on the Lyapunov theorem for such systems, a non-rigorous version of which can be stated as follows.

Theorem 2 (**Lyapunov theorem**) *A general time-varuing system described by the equation*

$$\dot{\bar{x}} = f(\bar{x}, t)$$

has the point x_e as a stable equilibrium point, if in a neighbourhood N_{x_e} of x_e there exists a continious scalar function $V(x,t)$ such that the folowing conditions are satisfied.

1. $V(x_e, t) = 0$ for all t

2. $V(x, t) \geq 0$ for all $x \neq x_e$ in the neighbourhood N_{x_e} for all t

3. $\dot{V}(x, t) \leq 0$ along the trajectories of the system in N_{x_e} for all t

If the above is true for the entire state-space, i.e. if N_{x_e} is the entire state-space, then x_e is *globally stable*. If condition (3) is satisfied as a strict inequality, i.e. $\dot{V}(x, t) < 0$ for all points other than the equilibrium, then the point x_e is *asymptotically stable.*

In our problem the system equations are given by Eq 3.1,3.2 and 3.3. After substitution of the control law given by Eq 3.13, we get the close-loop system as below

$$\dot{q}_0 = -(1/2)\,\bar{q}^T\bar{\omega} \tag{3.14}$$

$$\dot{\bar{q}} = (1/2)\,\Omega\bar{q} + (1/2)\,q_0\,\bar{\omega} \tag{3.15}$$

$$\dot{\bar{\omega}} = -\psi\bar{\omega} - \phi\bar{q} \tag{3.16}$$

The equilibrium point is obviously the target attitude

$$q_0 = 1\ ;\ \ q_1 = q_2 = q_3 = 0\ ;\ \ \omega = 0;$$

We can see that a simple positive scalar function of the instanteneous state that vanishes only at the equilibrium point can be given by

$$\begin{aligned} e &= (1/2)\bar{\omega}^T\bar{\omega} + (1-q_0)^2 + q_1^2 + q_2^2 + q_3^2 \\ &= (1/2)\bar{\omega}^T\bar{\omega} + 2(1-q_0) \end{aligned}$$

Suggested by this, we consider the following two similar Liapunov functions of the form

$$\begin{aligned} V_1 &= \tfrac{1}{2}\,\bar{\omega}^T\bar{\omega} + 2\phi(1-q_0) \\ \text{and}\qquad V_2 &= \tfrac{1}{2}\,\tfrac{1}{\phi}\,\bar{\omega}^T\omega + 2(1-q_0) \end{aligned}$$

with the restriction

$$\phi(t) > 0.$$

Hence, we have

$$\begin{aligned} V_1, V_2 &= 0 \quad \text{at the Eq. Point, for all } t \\ V_1, V_2 &\geq 0 \quad \text{for all other points, for all } t \end{aligned}$$

Differentiating V_1, and using the system equations 3.15 and 3.16, we have

$$\begin{aligned} \dot{V}_1 &= \bar{\omega}^T\dot{\bar{\omega}} - 2\phi\dot{q}_0 - 2\dot{\phi}(1-q_0) \\ &= \bar{\omega}^T\{-\psi\bar{\omega} - \phi\bar{q}\} - 2\phi\{-(1/2)\bar{\omega}^T\bar{q}\} + 2\dot{\phi}(1-q_0) \\ &= -\psi\,(\bar{\omega}^T\bar{\omega}) - \phi\bar{\omega}^T\bar{q} + \phi\bar{\omega}^T\bar{q} + 2\dot{\phi}(1-q0) \\ &= -\psi\,(\bar{\omega}^T\omega) + 2\dot{\phi}(1-q_0) \end{aligned}$$

Looking at the above expression , and noting that q_0 is always ≥ 0 . We conclude that the equilibrium point is globally asymptotically stable if the following set of conditions is satisfied

$$\begin{aligned} \phi &> 0 \\ \psi &> 0 \\ \text{and} \quad \dot{\phi} &\leq 0 \end{aligned}$$

Thus , the above is a set of *sufficient* conditions for global asymptotic stability, emerging from the first Liapunov Function V_1 . Now, differentiating V_2, we have

$$\begin{aligned} \dot{V}_2 &= \tfrac{1}{2}\cdot\tfrac{1}{\phi^2}\{\phi\cdot 2\bar{\omega}^T\dot{\bar{\omega}} - \dot{\phi}\bar{\omega}^T\bar{\omega}\} - 2\dot{q}_0 \\ &= \tfrac{1}{\phi}\cdot\bar{\omega}^T\dot{\bar{\omega}} - \tfrac{1}{2}\cdot\tfrac{\dot{\phi}}{\phi^2}\bar{\omega}^T\bar{\omega} - 2\{-\tfrac{1}{2}\bar{\omega}^T\bar{q}\} \\ &= \tfrac{1}{\phi}\bar{\omega}^T\{-\psi\bar{\omega} - \phi\bar{q}\} - \tfrac{1}{2}\tfrac{\dot{\phi}}{\phi^2}\cdot\bar{\omega}^T\bar{\omega} + \bar{\omega}^T\bar{q} \\ &= \tfrac{\psi}{\phi}\cdot\bar{\omega}^T\bar{\omega} - \bar{\omega}\,T\bar{q} - \tfrac{1}{2}\cdot\tfrac{\dot{\phi}}{\phi^2}\bar{\omega}^T\bar{\omega} + \bar{\omega}^T\bar{q} \\ &= -\tfrac{1}{2}\tfrac{1}{\phi^2}(2\phi\psi + \dot{\phi})\bar{\omega}^T\bar{\omega} \end{aligned}$$

From this we note that the equilibrium point is globally asymptotically stable if

$$(2\phi\psi + \dot{\phi}) \geq 0$$

A *sufficient* condition that ensures this, is

$$\phi\,,\,\psi > 0 \quad \text{with } \dot{\phi} \geq 0$$

Thus another set of *sufficient conditions*, emerging from the second Liapunov Function is

$$\begin{aligned} \phi &> 0 \\ \psi &\geq 0 \\ \text{and} \quad \dot{\phi} &\geq 0 \end{aligned}$$

Combining these two sets of conditions, each of which is sufficient to guarantee global stability, we see that as long as $\phi > 0$ and $\psi > 0$, both the cases of $\dot{\phi} \geq 0$ and $\dot{\phi} \leq 0$ ensures global asymptotic stability. Hence, the only required conditions sufficient to guarantee stability, are

$$\begin{aligned} \phi &> 0 \\ \psi &> 0 \end{aligned}$$

The control law proposed in [7] is clearly a special case of the above, with ϕ and ψ as positive scalars.

3.3 Other control laws

In the previous sections we looked into the form of the feedback controls that ensure that the spacecraft executes an Eigen-axis rotation. The form of control laws we arrived at were

$$u = \Omega J\omega - \psi J\bar{\omega} - \phi J\bar{q}$$

where $\psi(\cdot)$ and $\phi(\cdot)$ were scalar variables. We also obtained sufficient conditions on controls ϕ and ψ that ensure global asymptotic stability, in the case when they are time-varying. In general, however, ϕ and ψ can be scalar functions of the state-variables. In this section, a few more control laws of the above form are suggested

and tested. They are special cases of the above general form of control law, with ϕ, ψ, or both specified. Global asymptotic stability of the desired target point for these control laws is also demonstrated.

It may be mentioned that the purpose of suggesting the forms of ϕ and ψ in these control laws is primarily to illustrate the general nature of the above functions while achieving the Eigen-axis rotation. It illustrates the possibility of taking a wide choice of functions of the state variables as ϕ and ψ for achieving Eigen-axis rotation to the target attitude. The suggested control laws are examples from a possibly large number of control-laws that lead to an Eigen-axis rotation from any initial orientation to any target orientation.

In general, the system equations for the rigid-spacecraft rotations are of 7-dimentions, with actually 6 independant equations, the 1st of the quaternion kinematic equations being derivable from the other three, using

$$\bar{q}^T\bar{q} + q_0^2 = 1$$

But with control laws of the above form the body undergoes an Eigen axis trajectory, along which $\bar{q}$ and $\bar{\omega}$ are scalar multiples of the initial $\bar{q}(0)$ as given in Eq 3.4 and Eq 3.5. By substituting these in the close-loop Equations 3.14 3.15 and 3.16, we get the following equations

$$\dot{C}_q = \frac{1}{2}q_0 C_\omega = \frac{1}{2}C_\omega\sqrt{1 - Q^2C_q^2} \tag{3.17}$$

$$\dot{C}_\omega = -\psi\, C_\omega - \phi\, C_q \tag{3.18}$$

In this case the system equations get reduced to those of a two-dimentional system, with C_q and C_ω as the state variables. With ϕ and ψ remaining scalar, the above two-dimentional system is always equivalent to the 7-dimentional original system,

with the scalars C_q and C_ω being proportional to the magnitudes of the $\bar{q}$ and $\bar{\omega}$, whose directions have been determined by the initial attitude $\bar{q}(0)$. We can consider the above equations to be a two dimentiomnal system, with C_q and C_ω as the state variables, and $\phi(\cdot)$ and $\psi(\cdot)$ as the controls. Within this reduced system, $\phi(\cdot)$ and $\psi(\cdot)$ can be pure functions of time, in which case the reduced system is an 'open-loop' one, while the original system can be said to be a combined 'open-loop - closed-loop' one. On the other hand, if $\phi(\cdot)$ and $\psi(\cdot)$ are functions of $\bar{q}$ and/or $\bar{\omega}$, then the reduced system becomes a closed-loop controlled one, while the original system can be looked upon as one with a highly non-linear feedback.

With these observations, we proceed to suggest and analyse a few control laws.

3.3.0.1 Control law 2

A control law is suggested by taking ϕ and ψ as below.

$$\phi(\cdot) = q_0/2 \;\; ; \;\; \psi(\cdot) > 0$$

With the above control law the general close-loop system becomes,

$$\dot{q_0} = -(1/2)\,\bar{q}^T\bar{\omega} \tag{3.19}$$

$$\dot{\bar{q}} = (1/2)\,\Omega\bar{q} + (1/2)\,q_0\,\bar{\omega} \tag{3.20}$$

$$J\dot{\bar{\omega}} = -\psi\,J\bar{\omega} - \frac{1}{2}\,q_0\,J\,\bar{q} \tag{3.21}$$

or

$$\dot{\bar{\omega}} = -\psi\,\bar{\omega} - \frac{1}{2}\,q_0\,\bar{q} \tag{3.22}$$

The equilibrium point is obviously,

$$\bar{q} = \bar{\omega} = 0$$

The stability of the equilibrium point is studied with the following Lyapunov function.

$$
\begin{aligned}
V &= \tfrac{1}{2}\bar{\omega}^T\bar{\omega} + \tfrac{1}{2}\bar{q}^T\bar{q} \\
&= 0 \quad \text{atEq.}(\bar{q} = 0\ \bar{\omega} = 0 \\
&\geq 0 \quad \text{for all other points}
\end{aligned}
$$

Now differentiating this, and using Eqs 3.20 and 3.21, we have

$$
\begin{aligned}
\dot{V} &= \bar{\omega}^T\dot{\bar{\omega}} + \bar{q}^T\dot{\bar{q}} \\
&= \bar{\omega}^T\{-\psi\bar{\omega} - \tfrac{1}{2}q_0\bar{q}\} + \bar{q}^T\{\tfrac{1}{2}\Omega\bar{q} + \tfrac{1}{2}q_0\bar{\omega}\} \\
&= -\psi\bar{\omega}^T\bar{\omega} - \tfrac{1}{2}q_0\bar{\omega}^T\bar{q} + 0 + \tfrac{1}{2}q_0\bar{q}^T\bar{\omega} \\
&= -\psi\,\bar{\omega}^T\bar{\omega} \\
&< 0 \quad \text{for all points other than the Equilibrium}
\end{aligned}
$$

Thus the system is globally assymptodically stable, with respect to the point $\bar{q} = \bar{\omega} = 0$ Simulation results show that with suitable choice of ψ or ρ^2, the above control law achieves Eigen-axis rotation to the target point reasonably fast,with amount of energy spent being also reasonable - being comparable to that spent for low constant values of ϕ and ψ. The only considerable drawback of this form of ϕ is that for initial displacement of $\theta = 180^0$ exactly , in which case , $q_0 = \cos 90 = 0$, along with $\bar{\omega} = 0$, the control torque becomes zero, and the system fails to move. In other words,

$$
\begin{aligned}
q_0 &= 0 \\
\bar{\omega} &= 0
\end{aligned}
$$

is another equbilibrium point, although unstable . Hence, even for angles very close to 180^0 , the body returns to the target equilibrium point satisfactorily.

3.3.0.2 Control law 3

In this control law, we take ϕ and ψ as below.

$\phi(.) = (1 + \gamma\bar{\omega}^T\bar{q})$; $\psi(.) > 0$

Another form of the functions that was tried was the above, where γ is a $+ve$ scalar. We consider the lyapunov function

$$\begin{aligned} V &= (1/2)\bar{\omega}^T\bar{\omega} + 2(1-q_0) \\ &= 0 \quad \text{atEq.}\,(\bar{q}=0\;\bar{\omega}\;=0) \\ &\geq 0 \quad \text{for all other points} \end{aligned}$$

The equality occuring only at $\bar{\omega} \;=\; \bar{q} \;=\; 0$.

Differentiating , and using the ( ) , we have

$$\begin{aligned} \dot{V} &= \bar{\omega}^{\,T}\dot{\bar{\omega}} \;-\; 2\,\dot{q}_0 \\ &= \bar{\omega}^{\,T}\{-\,\psi\bar{\omega} \;-\; (1+\gamma\bar{\omega}^T\bar{q})\bar{q}\} \;-\; 2\{-\tfrac{1}{2}\bar{\omega}\;\bar{q}\} \\ &= -\psi\bar{\omega}^{\,T}\bar{\omega} \;-\bar{\omega}^{\,T}\bar{q} - \gamma(\bar{\omega}\;\bar{q})^2 + \bar{\omega}\;\bar{q} \\ &= -\,\psi\,\bar{\omega}^{\,T}\bar{\omega} \;-\; \gamma\,(\bar{\omega}^{\,T}\bar{q})^2 \\ &< 0 \quad \text{for all points other than the Equilibrium} \end{aligned}$$

Thus the system is stable, globally and assymptodically, with respect to the point

$$\bar{\omega} = 0\;;\; \bar{q} = 0\;;\; q_0 = 1$$

at the same time achieving Eigen-axis rotation, which is verified by simulations.

3.4 Robustness

The robustness of the above suggested control law, was checked by simulations, through variations in the inertia matrix used for feedback. Sufficient robustness was observed in all of them, for considerable variations. The nature of robustness for one of the control laws can be seen by the following analysis. Considering the control law with $\phi = q_0/2$, we have

$$u \;=\; \Omega J'\bar{\omega} \;-\; \psi J'\bar{\omega} \;-\; \phi J'\bar{q}$$

where J' is the inertia matrix with some error. Considering the same Lyapunov function as in control law 2,

$$V = \frac{1}{2}\bar{\omega}^T\bar{\omega} + \frac{1}{2}\bar{q}^T\bar{q}$$

we have

$$\begin{aligned}
\dot{V} &= \bar{\omega}^T\dot{\bar{\omega}} + \bar{q}^T\dot{\bar{q}} \\
&= \bar{\omega}^T J^{-1}\cdot J\dot{\bar{\omega}} + \bar{q}^T\dot{\bar{q}} \\
&= \bar{\omega}\, J^{-1}\{-\Omega J\bar{\omega} + \Omega J'\bar{\omega} - \psi J'\bar{\omega} - \tfrac{1}{2}q_0 J'\bar{q}\} \\
&\quad +\bar{q}^T\{\tfrac{1}{2}\Omega\bar{q} + \tfrac{1}{2}q_0\bar{\omega}\} \\
&= \bar{\omega}\, J^{-1}\Omega\,(J'-J)\,\bar{\omega} - \psi\bar{\omega}^T(J^{-1}J')\bar{\omega} - \tfrac{1}{2}q_0\bar{\omega}^T J^{-1}J'\bar{q} \\
&\quad + 0 + \tfrac{1}{2}q_0\bar{q}^T\bar{\omega} \\
&= \bar{\omega}^T J^{-1}\Omega\,(J'-J)\,\bar{\omega} - \psi\cdot\bar{\omega}^T(J^{-1}J')\,\bar{\omega} - \tfrac{1}{2}q_0\bar{\omega}\,(J^{-1}J'-I)\,\bar{q}
\end{aligned}$$

If the error in measuring J, i.e. the difference between J and J', is not unusually large, then both $(J'-J)$ and $(J^{-1}J'-I)$ are sufficiently small, and $J^{-1}J'$ is near about $I_{3\times3}$. At least, $(J'-J)$ and $(J^{-1}J'-I)$ are considerably smaller than $J^{-1}J'$. Hence the middle term in the above expression dominates. Morever, for not large errors between J and J', $(J^{-1}J'-I)$ is $+ve$ definite, making the expression for $\dot{V}$ $-ve$,thereby guaranteeing stability. Even for considerable errors, simulations often show stability of the system. However, it was seen that although stability of the equilibrium point remained unaffected, considerable errors in the inertia matrix, mainly in the diagonal elements, led to deviations from the Eigen-axis trajectory.

3.5 Conclusion

The analysis in this chapter led to a particular form of feedback laws that are capable of taking a spacecraft from any attitude to any other desired target attitude through an Eigen-axis rotation. Within this form, the possibility of having several particular

control laws were also illustrated, through three specific examples. That these control laws achieve the above objective have been clear from the mathematical analysis. Their verification with a computer simulation of the control system and its response for the control laws are done in the next chapter.

Chapter 4

Results and Discussions

The main objective of this thesis was to investigate the nature of control laws that can achieve an Eigen-axis rotation, from any arbitrary orientation to any other target orientation. The emphasys has been given towards highlighting the possibilty of generalising such control laws mentioned in previous literature. The control laws studied in this thesis were simulated on a MATLAB package suitable for efficient numerical processing. The results of these are presented in this chapter.

4.1 The control laws

It may be recalled that Eq , , and , gave the control system equations, with u being substituted by the proposed control law of the form

$$u = \Omega J\bar{\omega} - \psi(\cdot)\bar{\omega} - \phi(\cdot)\bar{q}$$

The control laws analysed may be recalled here.

1. $\phi(t)\ ,\ \psi(t)\ >\ 0$

2. $\phi = q_0/2\ ;\ \psi(\cdot) > 0$

3. $\phi = (1 + \gamma\bar{\omega}^T\bar{q})\ ;\ \psi(\cdot)\ >\ 0$

4.2 Response parameters

Simulations were carried out for the derived control laws, for a wide range of initial attitudes given in terms of their Eigen-axis $\hat{n}$ and the angle of Eigen-axis rotation θ. In each of the above control laws, a few different simple forms of $\psi(\cdot)$ were considered. The feature that was common to all the responses was that all of them followed the Eigen-axis rotation, as expected. This was verified by checking the value of the cross-product between $\bar{q}(0)$ which is coincident with the Eigen-axis for the initial attitude, and the instanteneious quaternion vector $\bar{q}$. For Eigen-axis rotation, these two vectors have to collinear throughout and hence their cross-product have to zero for all instants, which was verified. Apart from this, the other response parameters that were observed were the settling time, and nature of response in terms in terms of oscillatory or non-oscillatory. For a comparative observation, in some cases a measure of the total Energy involved in the manuever was also considered. This was taken as the integral of the body's Kinetic Energy through the manuever given by

$$\frac{1}{2}\int_0^{t_f} \bar{\omega}^T J\bar{\omega} \cdot dt$$

where $\bar{\omega}^T J\bar{\omega}$ is the body's Kinetic energy and t_f the final time.

4.3 General observations

The control law do not impose any condition on $\psi(\cdot)$ except that it should be positive. In the simulations $\psi(\cdot)$ was taken as function of time alone, which is nevertheless not necessary in control laws (2) and (3). It was observed that for cases where ψ rises faster compared to ϕ, the angular rate feedback tends to dominate the feedback after the first few seconds. Hence towards the end, with little attitude feedback in terms of the quaternion vector but high rate feedback, the body tends to come to a stop fast even before reaching the desired attitude $(1, 0, 0, 0)$. However, if ψ is taken as a positive

but decreasing function, the problem does not seem to arise. Another important fact that should be noted is that ϕ and ψ leaves sufficient scope for shaping the trajectory. For various types of these functions chosen, the nature of the response may vary over a wide range. Simulations showed a number of cases in which the response was highly oscillatory although allways remaining on the Eigen-axis rotation trajectory.

4.4 Results

4.4.1 Control law 1

This control law

$$u = \Omega J\bar{\omega} - \phi(t)\bar{q} - \psi(t)\bar{\omega}$$

contains both ϕ and ψ as $+ve$ functions of time. The following two typical cases are presented.

(a) $\phi(t) = 0.5t \ ; \ \psi(t) = t$

The above control law was tested with a number of initial attitudes of the body. The response in one such case is presented in Figs 4.1 and 4.2.

The initial attitude in this case was such that it could be reached from the reference orientation through a rotation of $\theta = 70^0$ about the line $\hat{n}$ with direction ratios (2, -4, 3). This arbitrarily chosen attitude corresponds to the quaternion (0.8192 0.2130 -0.4260 0.3195). The response was smooth, with settling time around 25 sec. The cross-product of $\bar{q}$ and $\bar{q}(0)$ is shown along with the quaternions in figure 4.1 and was found to be zero throughout.

(b) $\phi = t \ ; \ \psi = 0.5t$ In this case the initial attitude was taken as that reached by a rotation of 100^0 from the reference attitude about the line (6 3 -80) which corresponded to a quaternion of (0.6428 0.4402 0.2201 -.5870). The

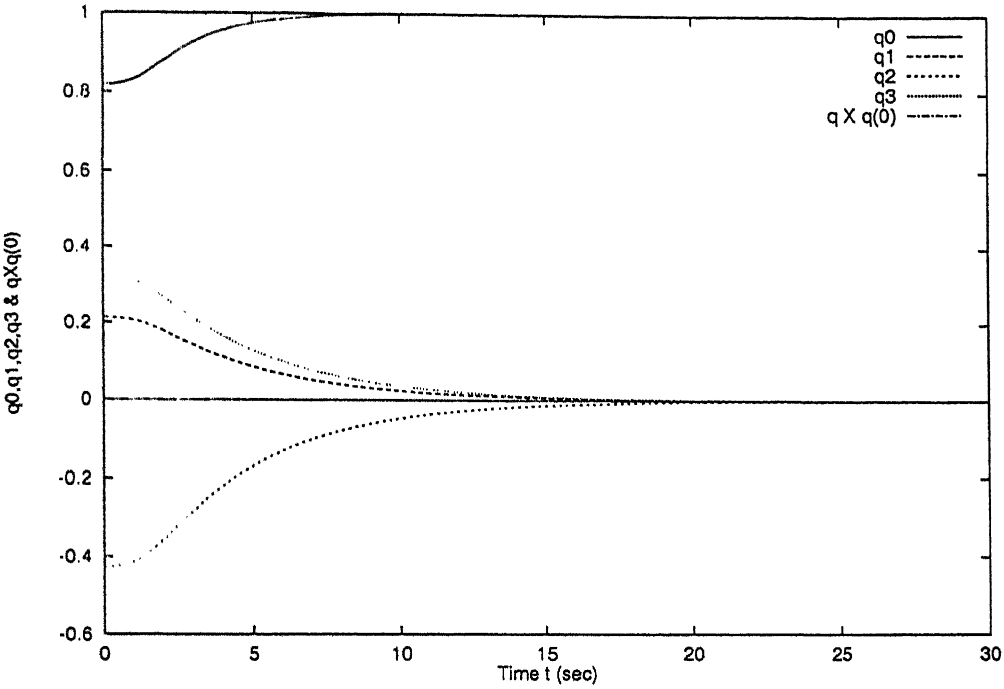

Figure 4.1: Control law 1(a) attitude

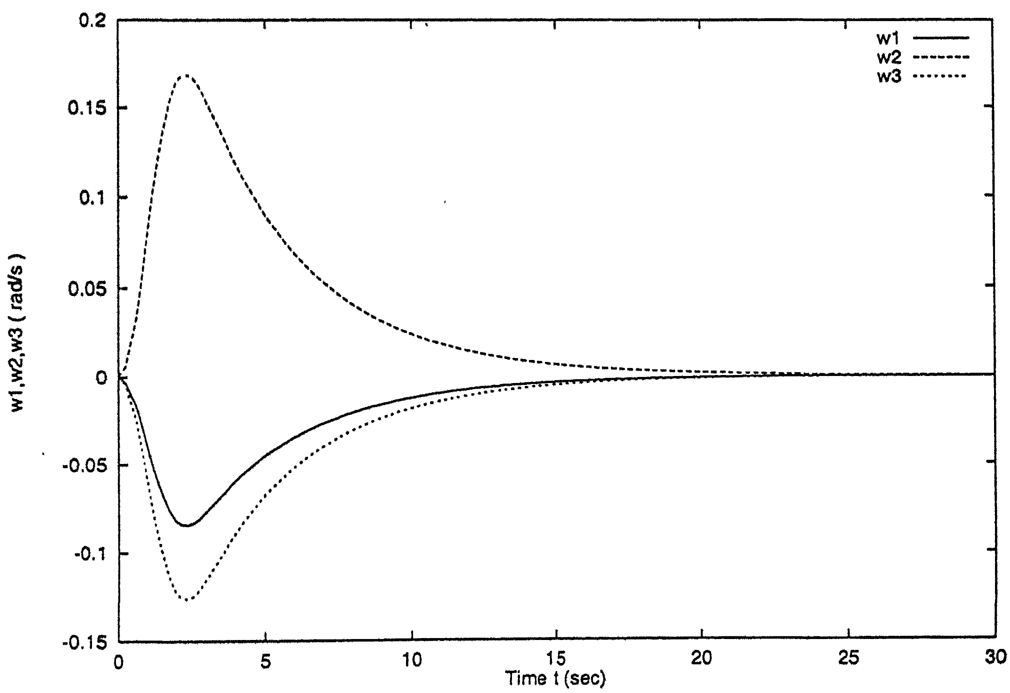

Figure 4.2: Control law 1(a) ang.vel

response is shown in Figs 4.3 and 4.4.

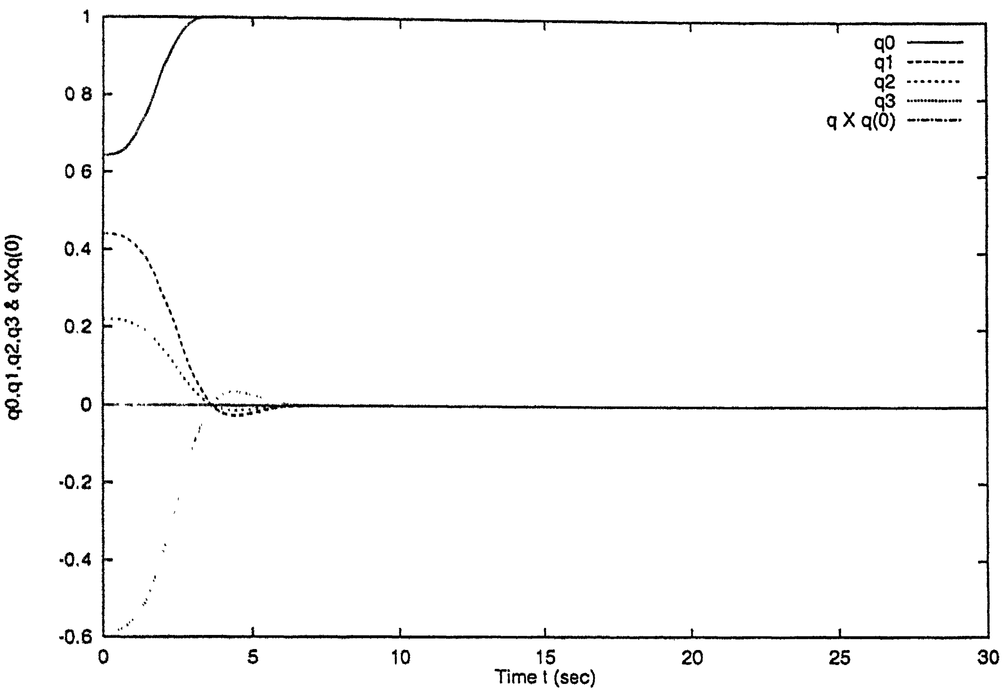

Figure 4.3: Control law 1(b) attitude

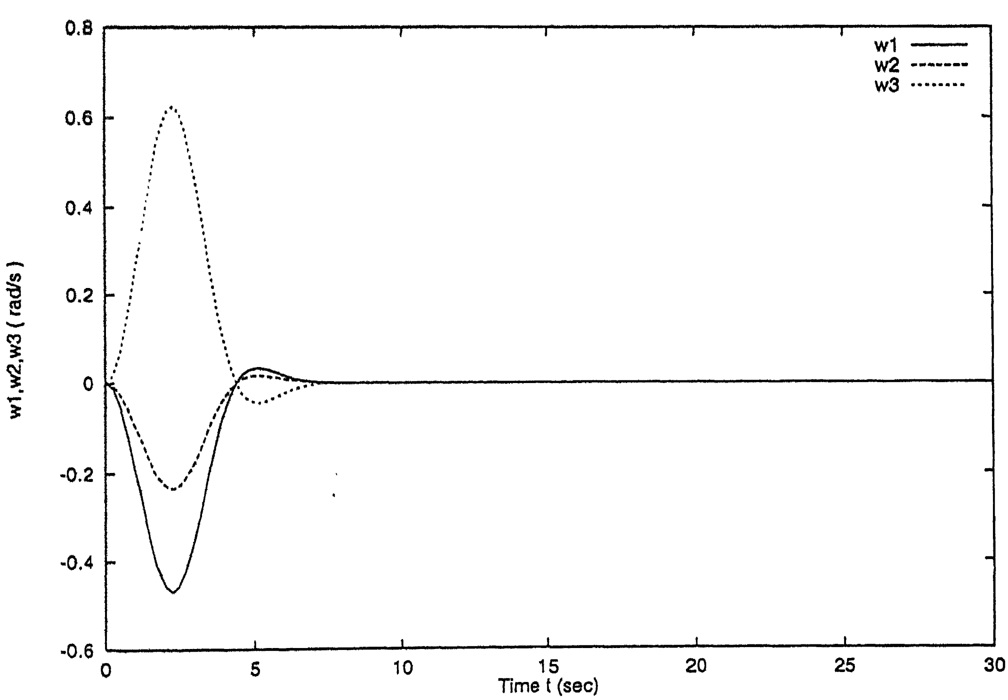

Figure 4.4: Control law 1(b) ang.vel

4.4.2 Control law 2

The control law is

$$\phi = \frac{1}{2}q_0 \quad ; \quad \psi > 0$$

This control law has ψ as a $+ve$ function. We present here two simple cases.

(a) $\psi = 0.9$

This is a case of constant ψ with ϕ as a function of the quaternion. The initial condition cosidered corresponded to the quaternion (0.9239 -0.3146 0.2002 0.0858) The response is shown in Fig 4.5 and 4.6. As can be seen, the response is smooth non-oscillatory with settling time about 15 sec. The cross-product of $\bar{q}(0)$ and $\bar{q}$ is again shown along with the quternion responses in Fig 4.5, and is found to be zero throughout.

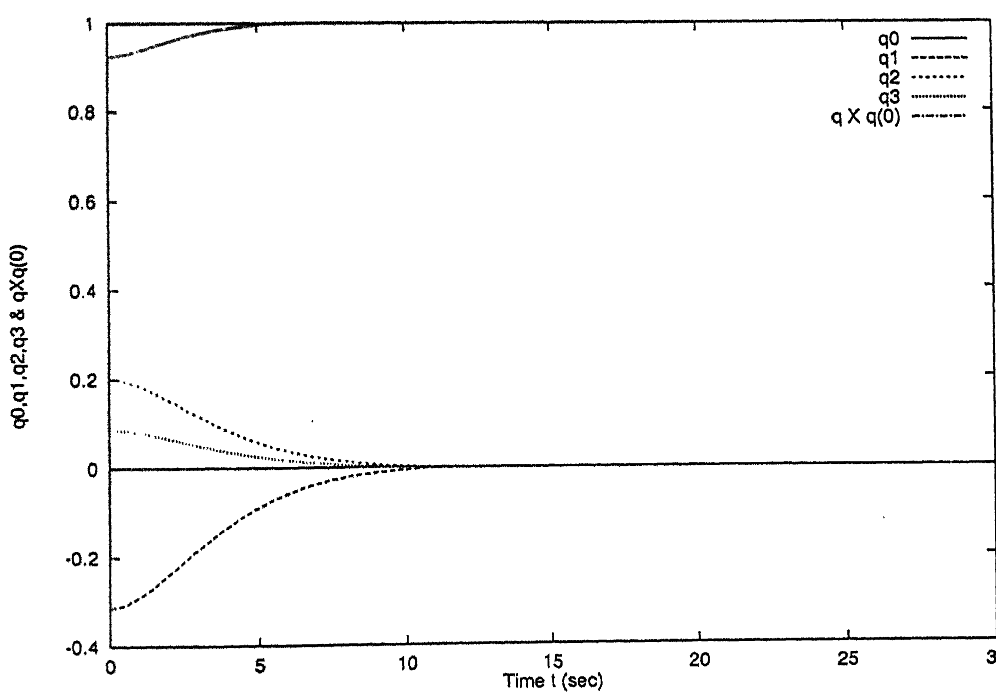

Figure 4.5: Control law 2(a) attitude

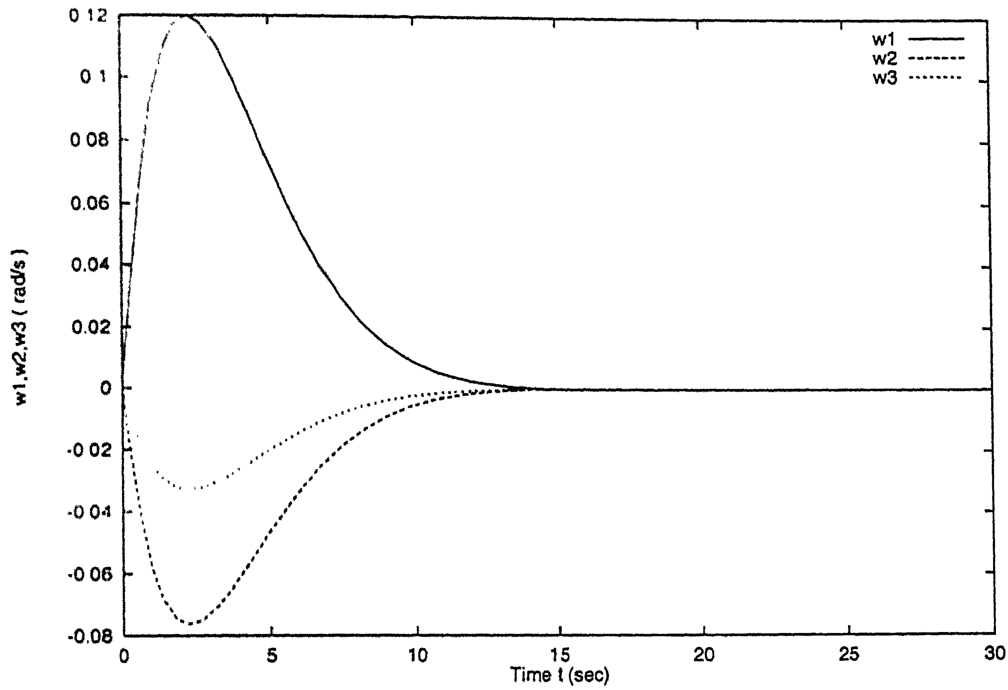

Figure 4.6: Control law 2(a) ang.vel

(b) $\psi = 30/(t+1)$

This is a case where ψ is a decreasing function of time. The initial attitude corresponded to (0.7132 0.4047 0.5665 -0.0809). For this, the response is shown in Figs 4.7 and 4.8. The response was found to be smooth non-oscillatory. The time taken for settling was around 35 sec.

4.4.3 Control law 3

The control law is

$$\phi = 1 + \gamma \bar{\omega}^T \bar{q} \quad ; \quad \psi(\cdot).0$$

where γ is any positive scalar constant or function. $\psi(\cdot)$ can also be any positive scalar function. Evidently, there are a large number of possibilities. We consider the following two forms, whose responses to arbitrary initial conditions are presented.

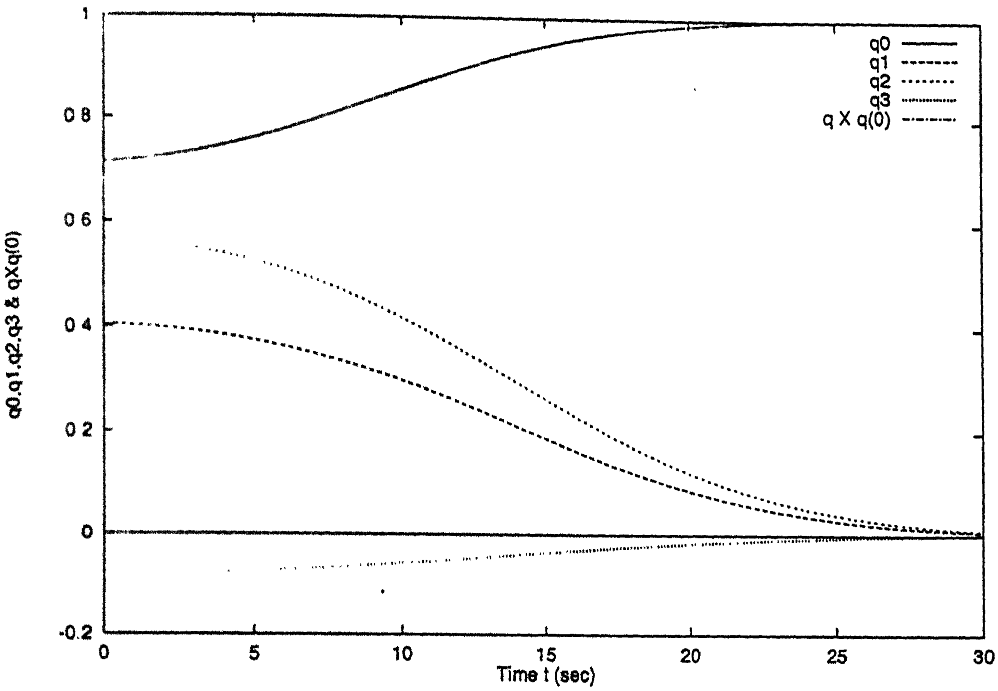

Figure 4.7: Control law 2(b) attitude

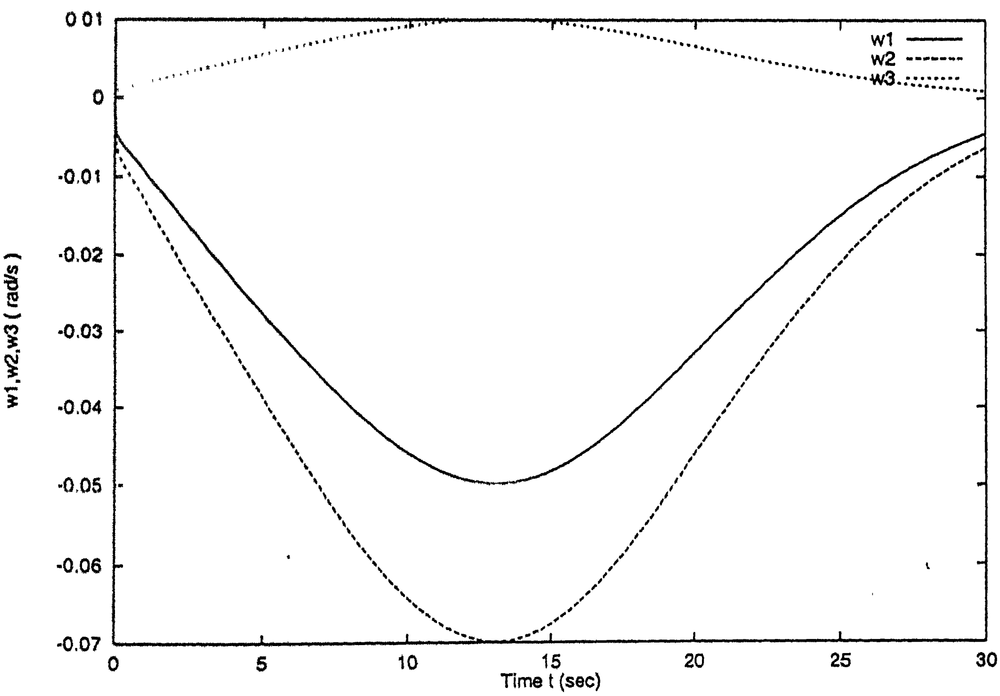

Figure 4:8: Control law 2(b) ang.vel

(a) $\phi = 1 + 0.5\,\bar{\omega}^T\bar{q}$; $\psi = t$

The initial attitude taken for the above is (0.6428 0.6221 0.4397 -0.0733). The response is shown in Figs 4.9 and 4.10.

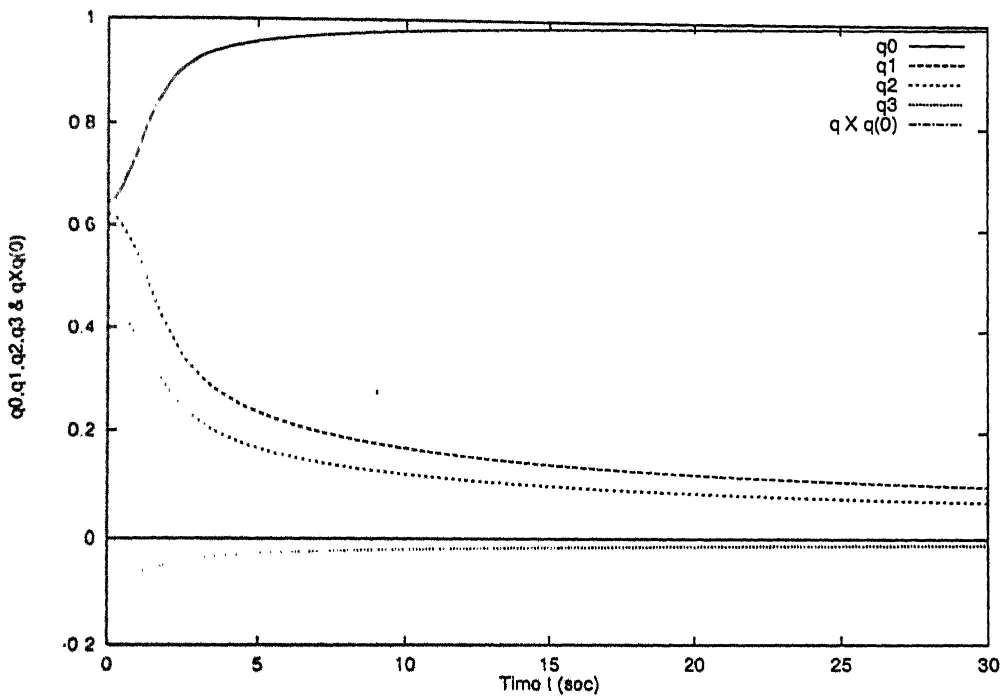

Figure 4.9: Control law 3(a) attitude

This is a case where towards the later stages, the angular rate feedback dominates the control law. Hence, it can be seen that towards the end , the approach of the body to (1, 0,0,0) slows down. The body almost comes to a stop at attitude slightly away from the desired (1, 0,0,0,). The response, however, is smooth and the body as usual traces the Eigen-axis rotation trajectory.

(b) $\phi = 1 + t \cdot \bar{\omega}^T\bar{q}$; $\psi(\cdot) = 1 + 50/(t+1)$

This is a case in which ψ has been taken as a decreasing function. The response is shown in Fig 4.11 and 4.12.

The problem of premature stopping is not seen here. The initial attitude arbitrarily chosen corresponded to a case of $\bar{n}$ having the direction cosines (3 9 -12) and θ being 175^0. This implied $q(0)$ =(0.4362 0.1959 0.5878 -0.7837). The response

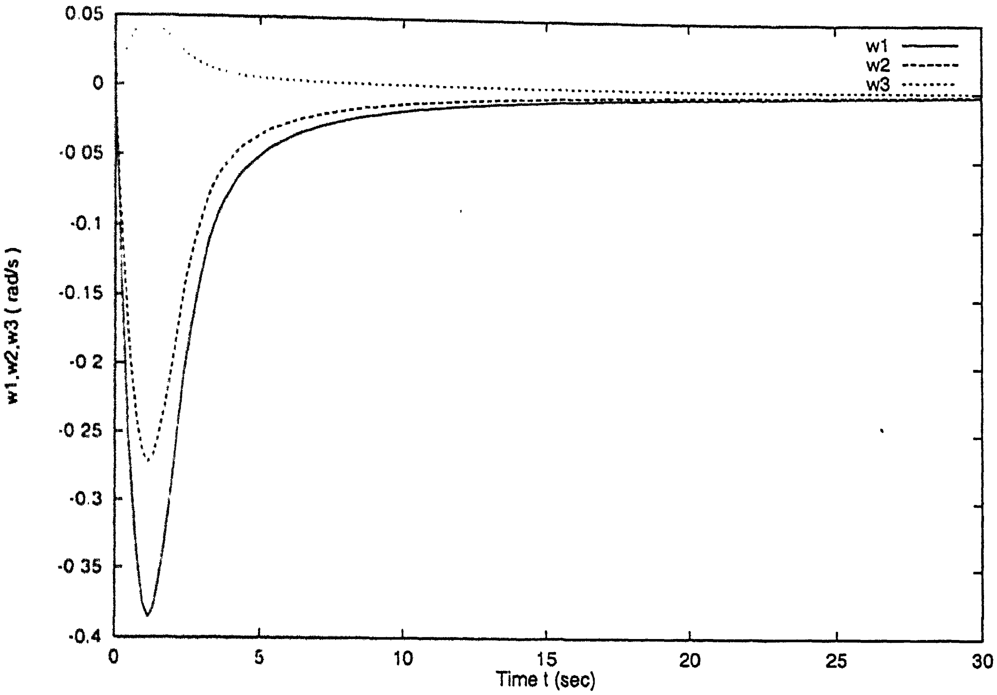

Figure 4.10: Control law 3(a) ang.vel

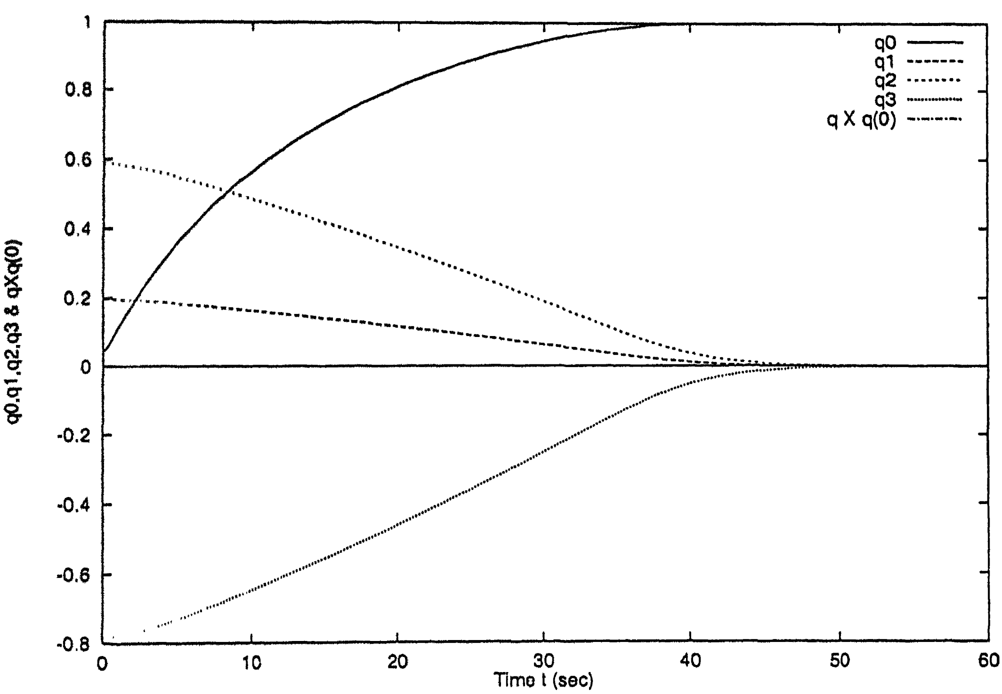

Figure 4.11: Control law 3(b) attitude

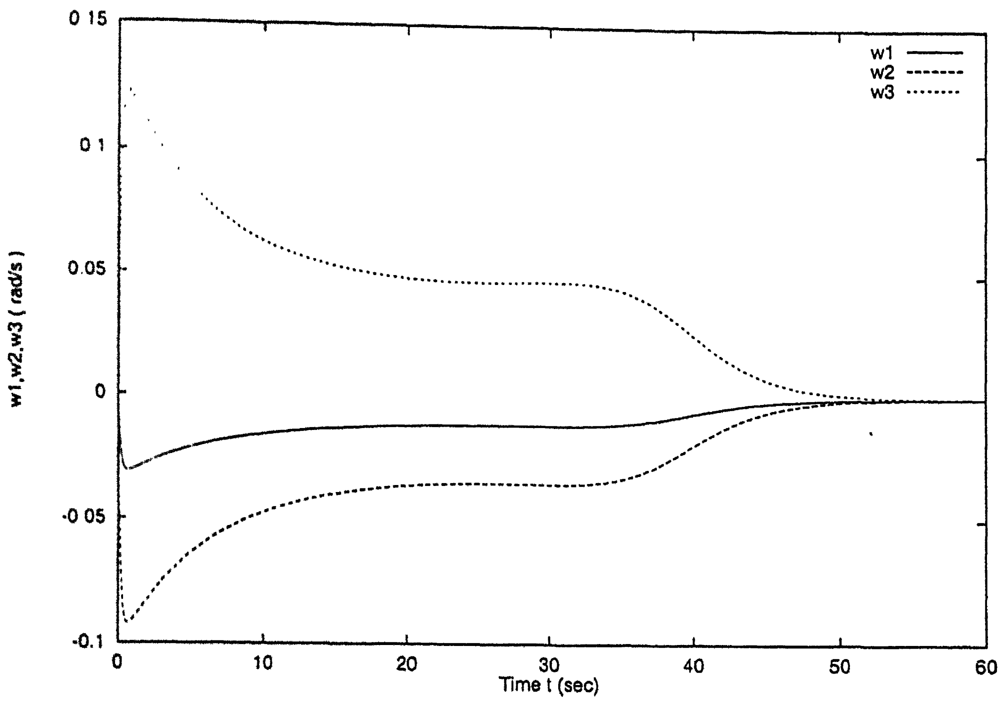

Figure 4.12: Control law 3(b) ang.vel

was smooth and the spacecraft settled completely by 52 sec at the desired attitude following Eigen-axis rotation.

4.4.4 A comparative view

For a rough comparison between the control laws, we considered the response of them for an arbitrarily chosen attitude corresponding to $\theta = 120^0$ and $\hat{n}$ having direction cosines (17 12 -2). For this $\bar{q}(0)$ =(0.5000 0.7043 0.4971 -0.0829) Below we give two cases, one each for Control laws 2 and 3. It should be noted that the conclusions reached may not be general, and may depend on the initial attitude.

Control law 2(c) :

$\phi = \frac{1}{2} q_0 \quad ; \quad \psi = 15/(t + 0.1)$

The responses for this are shown in Figs 4.13 and 4.14. The Kinetic Energy of the spacecraft during the maneuver is shown in Fig 4.15.

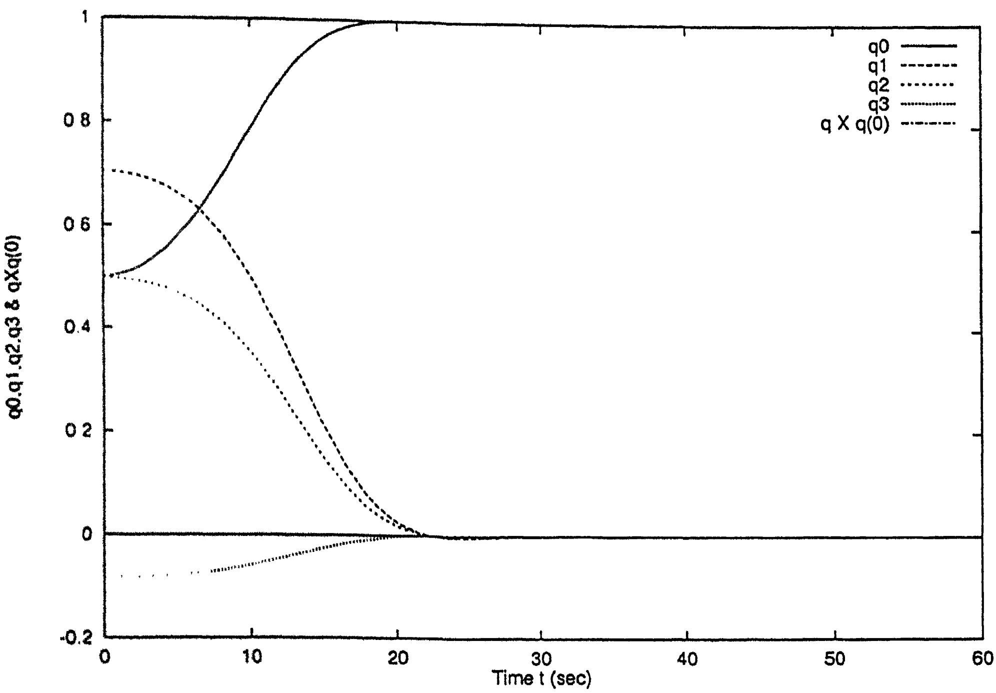

Figure 4.13: Control law 2(c) attitude

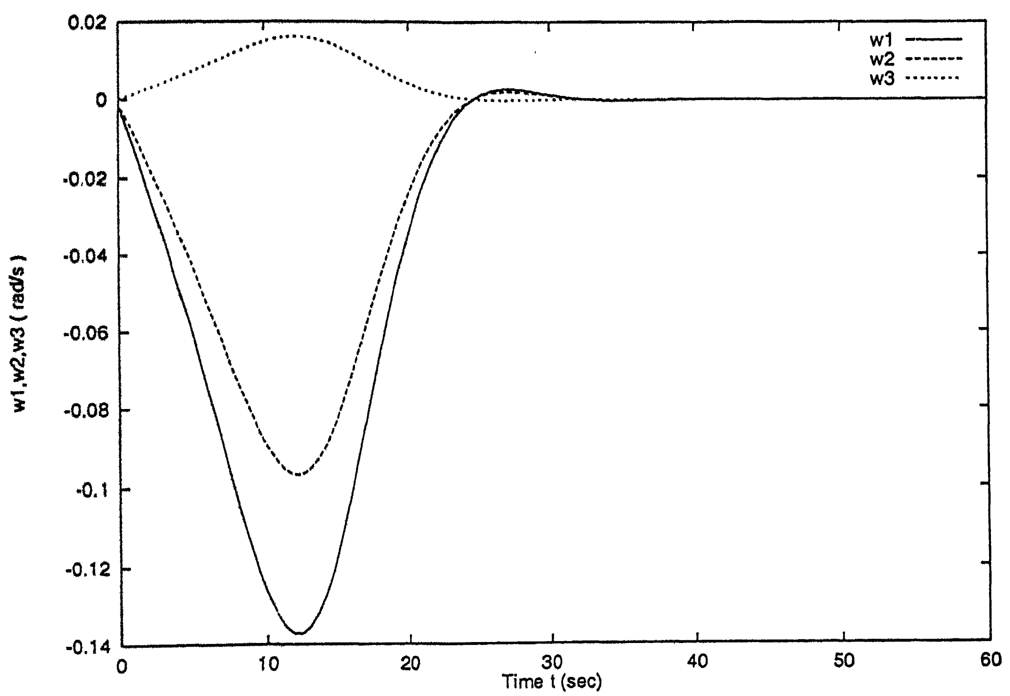

Figure 4.14: Control law 2(c) ang.vel

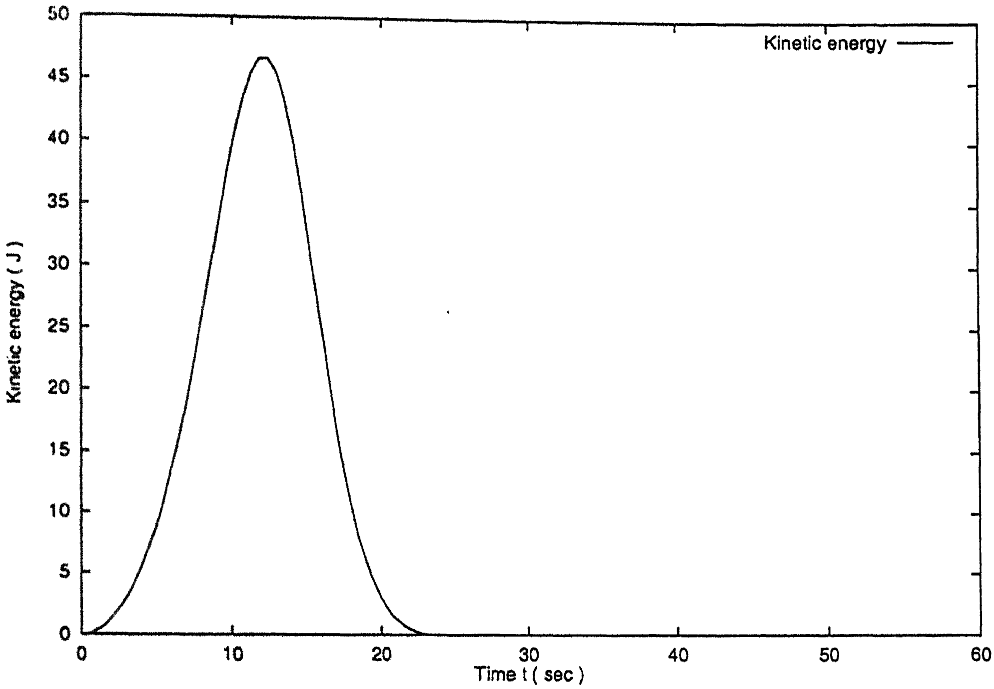

Figure 4.15: The kinetic energy for 2(c)

The integral of the kinetic energy over the entire maneuver given by $\frac{1}{2}\int_0^{t_f} \bar{\omega}^T J\bar{\omega}$ was found to be 700 J. In general, it was found that if the numerator of ψ which is 15 here is increased gradually this total kinetic energy over the maneuver decreases. However, the settling time gets increased.

Control law 3(c) :

$\phi = 1 + 0.5\bar{\omega}^T\bar{q} \quad ; \quad \psi = 10$

The responses for this is shown in Fig 4.16 and Fig 4.17. The variation of the Kinetic energy of the spacecraft through the maneuver is shown in Fig 4.18. The integral of thge kinetic energy for this case was found to be approximately 500 J.

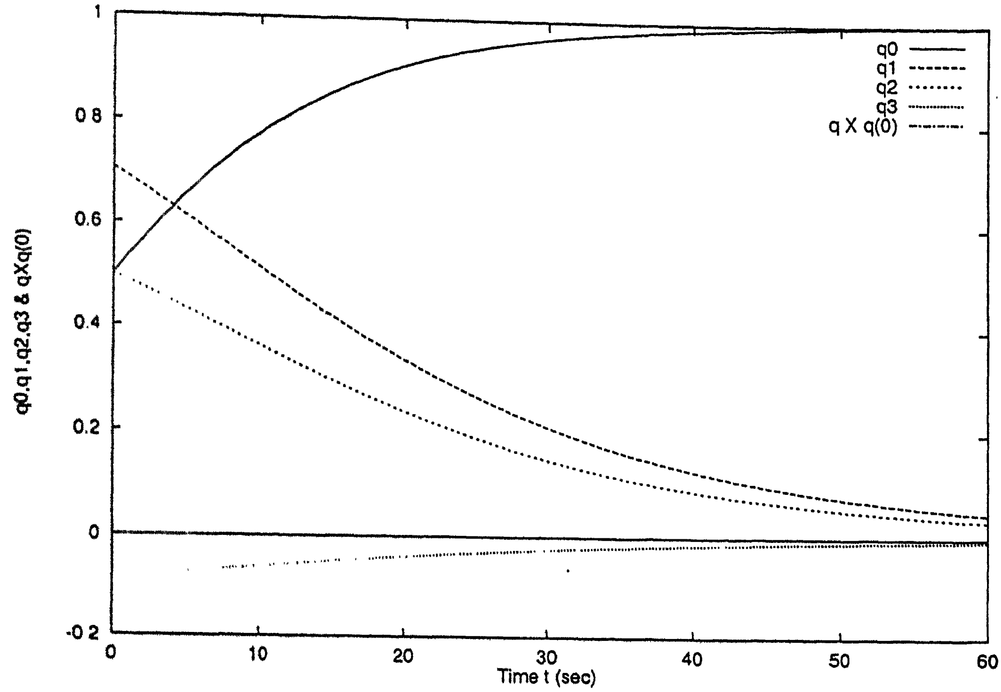

Figure 4.16: Control law 3(c) attitude

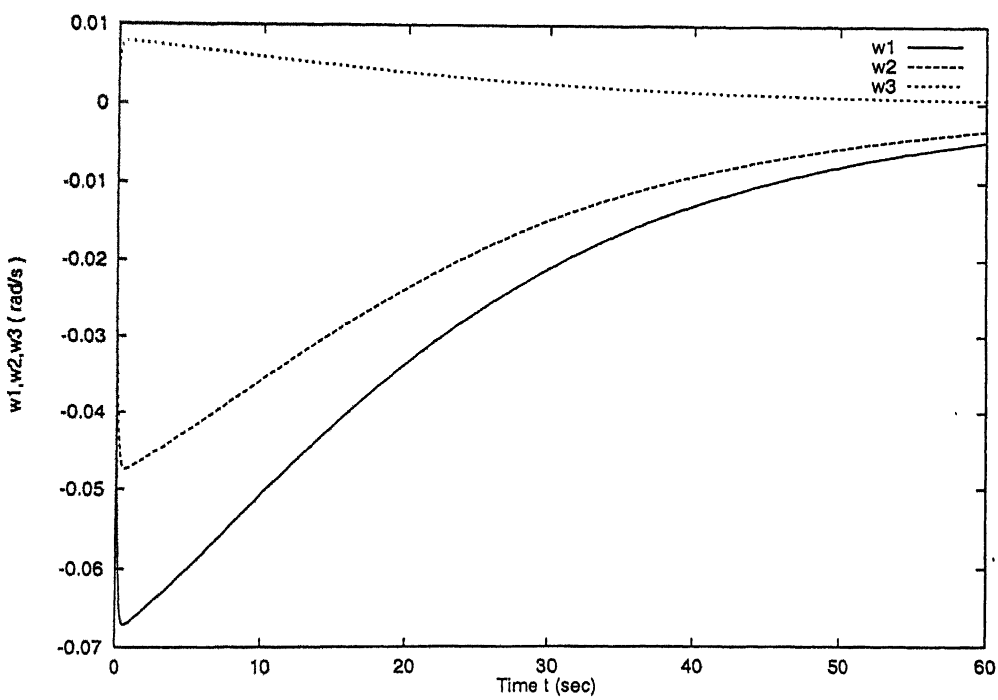

Figure 4.17: Control law 3(c) ang.vel

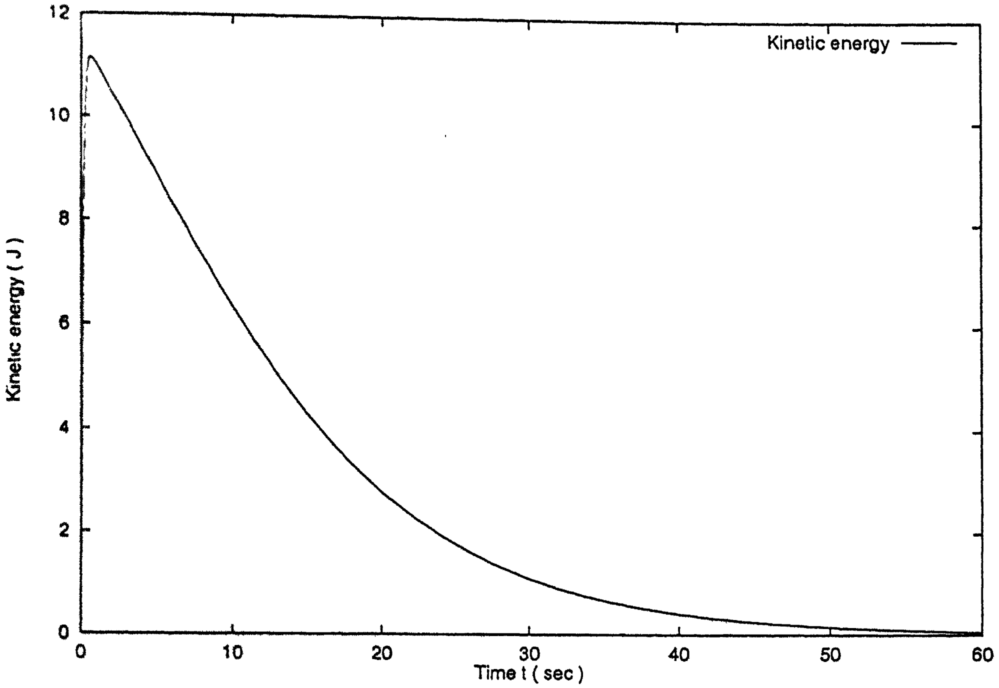

Figure 4.18: The kinetic energy for 2(c)

4.5 Conclusion

In this chapter the responses of the control system for the suggested control laws were determined and presented through computer simulations of the close-loop system. As is evident from the plots, all the suggested control laws were found to take the spacecraft to the desired equilibrium point or target attitude through an Eigen-axis rotation. The responses of the sustem were found to be of a varying nature ranging from highly oscillatory to highly smooth. However only a small section of all these possible responses could be presented. A more deeper understanding and improvement of the nature of the responses of these control laws may require further theoretical analysis of the system and the proposed laws.

Chapter 5

Future Work

In this thesis, it was found that a number of control laws are possible for achieving an Eigen-axis rotation from any attitude to any other target attitude. The generalization achieved opens up possibilities for future advancements for better understanding and use of the developed control laws. Three main areas where future work can concentrate can be identified as folows.

1. The conditions on ϕ and ψ that should be chosen for a particular shape of the $\bar{q}$ and $\bar{\omega}$ trajectory along the Eigen-axis, like non-oscillatory and smooth. This requires a deeper study into the relation between ϕ and ψ and the solutions of the close-loop Control System equations.

2. Among the diverse range of ϕ and ψ that can be used, an optimal choice with respect to some criteria remains to be found. An optimal-control analysis of ,the close-loop system with ϕ and ψ as inputs is a promising possibility.

3. A further investigation into the more general and complex form that arose during the development of the control law in section 3.1.3 . This may enable enhanced control over the trajectory shape within Eigen-axis rotation.

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Appendix A

Appendix

A.1 Some facts about rotations

In a 3-dimentional space, the number of independent variables required to specify the position and orientation of a rigid body is 6, which implies 6 degrees of freedom . For only rotation, without any translation, the degree of freedom is 3, the remaining 3 having been lost in fixing the center or origin of rotations.

Now, let x be the position vector of any arbitrary point with respect to some reference frame. For attitude analysis, any rotation of the body can be viewed from two frame-works. In the *active* frame-work, the body is considered to be rotated, with the axes or reference frame remaining fixed. In the *passive* frame-work, the body is considered to be fixed, while a reverse rotation of the axes is considered. In either case, the above mentioned point has a new position vector say x' which can be expressed as a linear transformation of x, given by $x' = Ax$ where A is 3×3 matrix. For pure rotations, however, the length of the vector must be preserved. Hence,

$$\begin{aligned} x'^T x' &= x^T x \\ \Rightarrow \qquad (Ax)^T(Ax) &= x^T x \\ \Rightarrow \qquad x^T A^T A x &= x^T x \\ \Rightarrow \qquad x^T(A^T A - I)x &= 0 \end{aligned}$$

As this must be true for *any* x, we get the important relation

$$A^T A = I \tag{A.1}$$

This is the orthogonality condition, obeyed by any rotational transformation matrix A. It is to be noted that this gives rise to six Equations involving the nine elements of the 3×3 matrix A, thus leaving only three of its elements totally independent. This is consistent with the three degrees of freedom for rotations arrived at earlier.

Now, we consider a reference-frame fixed in inertial space, calling it the *inertial* reference frame, and another *body-fixed* reference frame attached to the body, Then by the attitude of the body we mean the relative orientation of these two frames. The attitude of the body for which the two frames coincide, we will call its *reference attitude*. The attitude can be expressed in terms of, say, the angles between the axes of the two frames, or any other similar relevant quantity. The orthogaonal matrix A transforms the position vector (x_1, x_2, x_3) of a point on the body in the inertial- reference frame to that in the body-fixed frame, (x'_1, x'_2, x'_3). Hence, this is a representation of the body's attitude. This transformation matrix equivalently termed here as rotation matrix, contains as its elements the cosines of the angles between the axes of the two reference-frames, and is thus also called the Direction-Cosine Matrix.

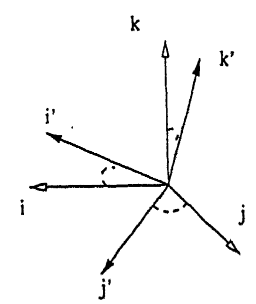

Figure A.1: The two reference frames

This can be shown as follows . We have

$$\begin{aligned} i &= (i \cdot i')i' + (i \cdot j')j' + (i \cdot k')k' \\ &= \alpha_1 i' + \beta_1 k' + \gamma_1 k' \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= \text{Cosine of angle between i, \& i}' \\ \beta_1 &= \text{Cosine of angle between i, \& j}' \\ \gamma_1 &= \text{Cosine of angle between i, \& k}' \end{aligned}$$

Similarly

$$\begin{aligned} j &= (j \cdot i')i' + (j \cdot j')j' + (j \cdot k')k' \\ &= \alpha_2 i' + \beta_2 j' + \gamma_2 k' \end{aligned}$$

and

$$\begin{aligned} k &= (k \cdot i')i' + (k \cdot j')j' + (k \cdot k')k' \\ &= \alpha_3 i' + \beta_3' + \gamma_3 k' \end{aligned}$$

The above gives the relations between the unit-vectors of the two sets of axes. But as the vector itself is fixed, we have

$$\begin{aligned} x_1' i' + x_2' j' + x_3' k' &= x_1 i + x_2 j + x_3 k \\ &= x_1(\alpha_1 i' + \beta_1 j' + \gamma_1 k') \\ &\quad + x_2(\alpha_2 i' + \beta_2 j' + \gamma_2 k') \\ &\quad + x_3(\alpha_3 i' + \beta_3 j' + \gamma_3 k') \\ \Rightarrow \quad x_1' i' + x_2' j' + x_3' k' &= (x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3)i' \\ &\quad + (x_1\beta_1 + x_2\beta_2 + x_3\beta_3)j' \\ &\quad + (x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3)k' \end{aligned}$$

which implies

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Evidently, this is $x' = Ax$, with A having cosine-elements $\alpha_i, \beta_i \ldots$.

As an example, consider a rotation of the axes through an angle θ around the z-axis, that is, keeping the z-axis fixed. The Direction-Cosine Matrix for the above rotation can be obtained as below.

Here noting that k and k' are same, and that i, i', j, j' are all perpendicular to k and k', we have

$$\gamma_3 = 1 \quad \text{and} \quad \alpha_3, \beta_3, \gamma_1, \gamma_2 = 0$$

Also,

$$(i \cdot i') = (j \cdot j') = \cos\theta \;\Rightarrow\; \alpha_1 = \beta_2 = \cos\theta$$

$$\text{and} \quad (i \cdot j') = (i' \cdot j) = \cos\theta \;\Rightarrow\; \beta_1 = \alpha_2 = -\sin\theta$$

Hence the Direction-Cosine Matrix for this case is

$$A_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations about the other axes have similar transformation matrices, and successive rotations may be represented by successive multiplications by matrices of these type. The resutant product matrices, which are the transformation matrices for the resultant attitudes, have elements as products of trigonometric quantities, and are thus numerically difficult to handle.
This calls for other attitude measures, and we shall study the quaternionic attitude measures, which are based on the important principle of Euler's theorem.

A.2 Euler's theorem for rigid bodies

One of the most important facts in rigid body motion, and the basis of the quaternionic representaion of attitude used in this thesis, is what is well-known in rigid-body mechanics as the *Euler's Theorem for rigid-bodies*. The theorem is stated as follows.

Theorem 3 (Euler's theorem) *The most general displacement of a rigid-body with one point fixed, is a rotation about some axis.*

For a proof of a theorem, two important results should be considered.

Result 1 *Any matrix A representing a rigid-body displacement with one point fixed, has at least one of its Eigen-values equal to 1 with a corresponding non-trivial Eigen-vector.*

Proof:
To analyse the motion of rigid-body with one point fixed, we consider two reference frames, as in previous section. One is is fixed to the body, another being fixed in inertial space. In addition, we consider the fixed point on the body as the common origin. As before, the attitude may be expressed as a linear transformation between the position vectors of a point on the body in the two frames, represented by $x' = Ax$. As the body is rigid, all position vectors have their length preserved. Through an analysis exactly as in previous section, A can be shown to be orthogonal. Hence,

$$A^T A = I$$

Now this implies,

$$\begin{aligned} \det A^T \cdot \det A &= (\det A)^2 &= 1 \\ \Rightarrow \qquad & \det A &= \pm 1 \end{aligned}$$

It can be shown that the case of -1 is valid for a class of transformations called *reflections*, in which a right-handed axes system is transformed to a left-handed one. But as we considered both body-fixed and inertial frames to be right-handed, and also coincident for the reference rotation, this transformation is not possible as long

as the body is rigid. Therefore in this case, the determinant is +1.
Further, for a general orthogonal matrix A, we have

$$\begin{aligned}
A^T A &= I \\
\Rightarrow \quad (A-I)A^T &= I - A^T \\
\Rightarrow \quad \det(A-I)\cdot\det A^T &= \det(I-A^T) \\
\Rightarrow \quad \det(A-I) &= \det(I-A^T)
\end{aligned}$$

using $\det A^T = \det A = 1$.
Therefore, this implies

$$\det(A-I) = \det(I-A) \tag{A.2}$$

But from theory of determinants, we have for A being 3×3

$$\det(I-A) = (-1)^{-3}\det(I-A)$$

Combining this with Eq A.2 we can write

$$\begin{aligned}
\det(A-I) &= -\det(A-I) \\
\Rightarrow \quad \det(A-I) &= 0
\end{aligned}$$

From this, it can be concluded that the system of linear equations

$$(A-I)x = 0$$

$$\text{or} \qquad Ax = x$$

has a non-trivial solution. This evidently implies that A has at least one Eigen-value equal to 1 and a corresponding non-trivial Eigen-vector x.

Result 2 *All the Eigen-values of any orthogonal matrix are of magnitude 1*

Proof: Let λ be an Eigen-vector of A. λ may be complex, or purely real. Hence, we have,

$$Ax = \lambda x$$

It also follows from elementary Linear Algebra that λ^* is a right Eigen-vector of A^T. That is,

$$x^T A^T = \lambda^* x^T$$

Therefore, multiplying the last two equations, and using Eq A.1 we get

$$\begin{aligned} x^T(A^TA)x &= \lambda^*\lambda \cdot x^Tx \\ \Rightarrow \qquad x^Tx &= \lambda^*\lambda \cdot x^Tx \end{aligned}$$

From the above, it is evident that

$$\lambda^*\lambda = \mid \lambda \mid^2 = 1$$

where $\mid \lambda \mid$ denotes the magnitude or modulus of λ. This proves the result.

Now we are in a position to give a proof of the Euler's theorem. From the above two results, it may be concluded that among the three Eigen-vectors, one has a value of 1, while the other two are in general complex numbers of magnitude 1. Further, if these two are not real, but complex, then they have to be conjugates. This is because A being real, its characteristic equation

$$\det(A - \lambda I) = 0$$

have real coefficients, implying that complex roots occur in conjugates.
Thus , in general the three Eigen-values can be written as

$$\lambda_{1,2,3} = 1 \ , \ e^{i\theta} \ , \ e^{-i\theta}$$

The Eigen-vector corresponding to the Eigen-value 1, is clearly a vector that does not get affected by the transformation, suggesting that it may be the axis of a rotation transformation.
The physical significance of the quantity θ can be understood if we take the three Eigen-vectors as our reference-frame, taking the axis Eigen-vector (that with Eigen-value 1) as the z-axis. The resultant transformation A' in this reference-frame is given by

$$A' = U^{-1}AU$$

where U is the matrix containing the 3 Eigen-vectors as its columns, 3rd column being the axis-Eigen-vector. It can be seen that A' is the same as the Direction-Cosine or Transformation matrix $A_{z,\theta}$ obtained for the case of a rotation about the z-axis, through an angle θ, in Sec A.1. Hence, it is clear that any 3×3 Real Orthogonal matrix representing a transformation of a rigid-body with one point fixed is equivalent to a rotation about an axis, through an angle, say θ. This is the Euler's theorem for rigid-bodies.

The axis of the rotation is the Eigen-vactor corresponding to the Eigen-value of 1, while the other two Eigen-values $e^{i\theta}, e^{-i\theta}$ give the angle of rotation θ. A special case is $\theta = 0$, in which case all Eigen-values are 1, and transformation is an identity

transformation which also can be viewed as a rotation through an angle $\theta = 0$. An easy way to find the angle θ through which a given A rotates the body can be found by observing that

$$trace[A] = \sum_{i=1}^{3} \lambda_i = 1 + 2\cos\theta \tag{A.3}$$

From which , θ is easily obtained as

$$\cos\theta = \frac{1}{2}(trace[A] - 1)$$

Hence, given any real orthogonal matrix A, the axis and angle for the corresponding rotation can be readily found.

A.3 Quaternions and attitude representation

At any instant, the attitude of a rigid-body may be considered as resulting from a motion, or series of motions, that has taken it from its reference attitude to the present one. Reference attitude, it may be recalled, is the attitude for which the body-fixed frame coincides with the inertial frame. Now, Euler's theorem implies that this motion or series of motions, is always equivalent to a single rotation about an axis through an angle. This axis, which is fixed with respect to the inertial frame for a given attitude, is called the *Eigen-axis* of that attitude. Therefore the Eigen-axis and the angle of rotation can serve to uniquely describe the rotation from the reference orientation, thereby describing the present orientation too. Denoting an unit vector along the Eigen-axis as $\hat{n}$, and the angle of rotation as θ, an algebraic quadruplet $q = (q_0, q_1, q_2, q_3)$ called quaternion is defined as below .

$$q_0 = \cos\theta/2 \tag{A.4}$$

$$\bar{q} = \hat{n}\,\sin\theta/2 \tag{A.5}$$

where $\bar{q} = (q_1, q_2, q_3)$ is considered as the vector part of q, and q_0 the scalar part. A quaternion with scalar part zero is termed a *vector quaternion.* A norm $N(q)$ of q is also defined as

$$N(q) = q_0^2 + \mid \bar{q} \mid^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

It can be checked that for the above definition of q in Eq A.4 and Eq A.5, the norm $N(q)$ is always 1. Thus they are precisely called unit quaternions.
A detailed discussion on the mathematical structure and algebra of quaternions in genaral is given in the section A.4.

A.3.1 Relation between rotation matrix and quaternions

For quaternions to be useful parameters for attitude representation, rotations also must be representable by them. This further implies that the attitude transformation matrix A must be expressible totally in terms of quaternion parameters. Here, the relation between A and the parameters of q are derived. However,in this case, for convenience, we chose the *active* framework. In this frame-work, — it may be recalled, the body is considered to have moved, with the frame fixed.

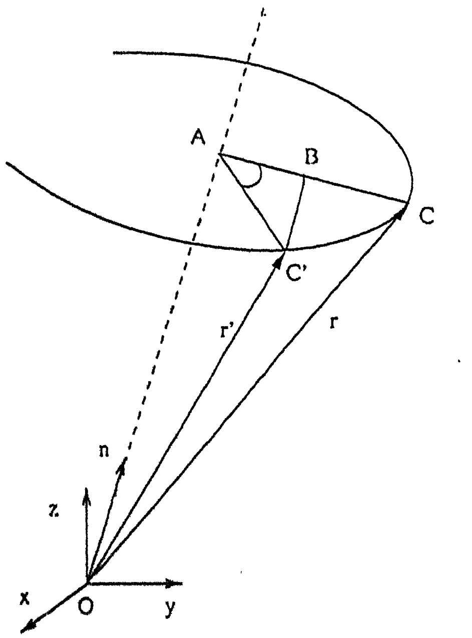

Figure A.2: Rotation of vector r about OA

In the figure A.2, a point C specified by the position vector OC, is rotated about an arbitrary axis OA, to the new location C', with position vector OC'. The vectors OC and OC' are denoted by $\bar{r}$ and $\bar{r}'$, while $\hat{n}$ is a unit vector along the axis OA. The angle through which OC rotates is θ.

From the figure, $| OA |= (\bar{r} \cdot \hat{n})$, which gives

$$OA = (\bar{r} \cdot \hat{n})\, \hat{n} \tag{A.6}$$

Again

$$| AC |= \bar{r} \times \hat{n} \tag{A.7}$$

Also, from the fig and Eq A.6,

$$AC = OC - OA = \bar{r} - (\hat{n} \cdot \bar{r})\, \hat{n} \tag{A.8}$$

Therefore, from Eqs A.7, and A.8, and noting that AC' and AC are of the same magnitude, we can write

$$\mid AC' \mid = \mid AC \mid = \mid \bar{r} \times \hat{n} \mid = \mid \bar{r} - (\bar{r} \cdot \hat{n})\, \hat{n} \mid \tag{A.9}$$

Now, noting the magnitude and direction of AB from the figure, we can write

$$AB = \mid AC' \mid \cos\theta \cdot \frac{AC}{\mid AC \mid}$$

which after using Eq A.8 and A.9 becomes

$$AB = \mid AC' \mid \cos\theta \cdot \frac{AC}{\mid AC \mid} = [\bar{r} - (\bar{r} \cdot \hat{n})\, \hat{n}]\, \cos\theta \tag{A.10}$$

Similarly, noting the magnitude and direction of BC', and using Eq A.9, we obtain

$$BC' = \mid AC' \mid \sin\theta \cdot \frac{\bar{r} \times \hat{n}}{\mid \bar{r} \times \hat{n} \mid} = (\, \bar{r} \times \hat{n}\,)\, \sin\theta \tag{A.11}$$

Finally, we express the rotated vector $\bar{r}'$ in terms of $\bar{r}$, the axis, and the angle of rotation by using the Eqs A.6, A.10, and A.11 as follows

$$\begin{aligned} \bar{r}' &= OA + AB + BC' \\ &= (\bar{r} \cdot \hat{n})\, \hat{n} + [\bar{r} - (\bar{r} \cdot \hat{n})\, \hat{n}]\, \cos\theta + (\, \bar{r} \times \hat{n}\,)\, \sin\theta \end{aligned}$$

which after rearrangement gives

$$\bar{r}' = \bar{r} \cos\theta + (\bar{r} \cdot \hat{n})\hat{n}[1 - \cos\theta] + (\, \bar{r} \times \hat{n}\,)\, \sin\theta \tag{A.12}$$

This is the well-known rotation-formula for finite rotations. At this stage, we introduce the quaternion, as defined in Eq A.4 and A.5. Rewriting the above two equations for convenience, we have

$$\begin{aligned} q_0 &= \cos\theta/2 \\ \bar{q} &= \hat{n}\, \sin\tfrac{\theta}{2} \end{aligned}$$

With these definitions, we can express the terms of the right-hand-side of the rotation formula Eq A.12 in terms of the quaternion parameters as below. We have

$$\bar{r} \cos\theta = \bar{r}\, (2\cos^2\theta/2 - 1) = \bar{r}\, (2q_0^2 - 1) = \bar{r}\, (q_0^2 - q_1^2 - q_2^2 - q_3^2)$$

Similarly,

$$(\bar{r} \cdot \hat{n})\, \hat{n}\, [1 - \cos\theta] = \frac{(\bar{q} \cdot \bar{r})\, \bar{q}}{\sin^2\theta/2}\, [2\sin^2\theta/2] = 2\, (\bar{q} \cdot \bar{r})\, \bar{q}$$

and

$$(\bar{r} \times \hat{n}) \sin\theta = \frac{(\bar{r} \times \bar{q})}{\sin\theta/2} \cdot 2\sin\theta/2 \cdot \cos\theta/2 = 2\,(\bar{r} \times \bar{q})\, q_0$$

With these substitutions, the rotation formula Eq A.12 becomes

$$\bar{r}' = \bar{r}\,(q_0^2 - q_1^2 - q_2^2 - q_3^2) + 2(\bar{q} \cdot \bar{r})\,\bar{q} + 2\,(\bar{r} \times \bar{q})\, q_0 \tag{A.13}$$

The above is a relation giving the rotated vector $\bar{r}'$ in terms of the old vector $\bar{r}$ and the quaternions corresponding to the particular rotation. If the position vectors are given in terms of (x, y, z) co-ordinates with respect to some inertial reference frame, then the above relation Eq A.13 can be written as a linear transformation of the co-ordinates. This is easily obtained by expanding Eq A.13.

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = 2 \begin{bmatrix} q_1^2 + q_0^2 - \frac{1}{2} & q_1 q_2 - q_3 q_0 & q_1 q_3 + q_2 q_0 \\ q_1 q_2 + q_3 q_0 & q_2^2 + q_0^2 - \frac{1}{2} & q_2 q_3 - q_1 q_0 \\ q_3 q_1 - q_2 q_0 & q_3 q_2 + q_1 q_0 & q_3^2 + q0^2 - \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{A.14}$$

where (x, y, z) and (x', y', z') are the co-ordinates of $\bar{r}$ and $\bar{r}'$ respectively, in the chosen inertial reference-frame. The above is clearly of the form $x' = A(q)x$, with $A(q)$ being the rotation-matrix in terms of q elements.

A.3.2 The reference attitude in quaternions

The reference- attitude has been defined as the one in which the body-fixed frame and the inertial frame coincide, and hence, x and x' are same. For this, ,obviously, $A = I$, where I is the 3×3 identity matrix. Thus, $A = I$ represents the reference attitude.

Let q_e be the quaternion representation of the reference attitude, then for this, $A(q_e) = I$. This is satisfied only for the case

$$q_0 = 1; q_1 = q_2 = q_3 = 0$$

Hence, the reference attitude in represented as $(1, 0, 0, 0)$ in quaternions. This is consistent physically, since this corresponds to the case of $\theta = 0$ in the defining equations of q, Eq A.4 and A.5, and any $\hat{n}$. This just means that the attitude is such that the body has rotated through zero angle about any axes, to arrive at it from the reference attitude. A source of ambiguity may be the fact that $(-1, 0, 0, 0)$ also makes $A(q) = I$ and hence is also a candidate for the reference attitude representation. This reflects the physical fact that the reference attitude may be considered as resulting from a 360^0 rotation from itself, about any axis. This makes $\theta = 360^0$, with any $\hat{n}$, and thus q_0 becomes $\cos 360^0 = -1$, with $\bar{q} = 0$, giving $(-1, 0, 0, 0)$ as the related quaternion.

A.4 Basic quaternion algebra

Quaternions constitute an algebra by means of which spacial kinematics, especially spherical kinematics can be treated elegantly. They were suggested by Hamilton, although in a slightly different form,but subsequently pushed to the background at the advent of vector-algebra. Towards the latter half of this century, interest in it was revived for its suitability in rotational kinematics. It is closely related to Euler Parameters, often termed equivalently, and is also similar to spinors in quantum mechanics.
A quaternion can be defined as complex number with four units, $1, i, j, k$ written as

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where $q_0, \ldots, q_3$ are real numbers. q_0 is called the scalar part and (q_1, q_2, q_3) the vector part. In fact, a quaternion with $q_0 = 0$ can be considered a 3-Dimentional Euclidean vector, and is called a vector quaternion. Accordingly, a quaternion can be written formally as the sum of a real scalar and a vector.

$$\begin{aligned} q &= q_0 + \bar{q} \\ \text{where} \quad \bar{q} &= q_1 + q_2 + q_3 \end{aligned}$$

The addition of quaternions is defined by

$$\begin{aligned} q + q' &= (q_0 + q_1 i + q_2 j + q_3 k) + (q'_0 + q'_1 i + q'_2 j + q_3 k) \\ &= (q_0 + q'_0) + (q_1 + q'_1)i + (q_2 + q'_2)j + (q_3 + q'_3)k \end{aligned}$$

Multiplication of quaternions is based on the following rules for the multiplication of the units .

$$1.i = i.1 = i \quad ; \quad 1.j = j.1 = j \quad ; \quad 1.k = k.1 = k$$

$$i^2 = j^2 = k^2 = -1$$

$$i.j = -j.i = k \quad ; \quad j.k = -k.j = i \quad ; \quad k.i = -i.k = j$$

In addition, it is distributive with respect to summation. With this, we have

$$\begin{aligned} qq' &= (q_0 + q_1 i + q_2 j + q_3 k)(q'_0 + q'_1 i + q'_2 j + q'_3 k) \\ &= (q_0 q'_0 - q_1 q'_1 - q_2 q'_2 - q_3 q'_3 - q_4 q'_4) + (q_0 q'_1 + q_1 q'_0 + q_2 q'_3 - q_2 q'_3)i \\ &\quad +(q_0 q'_2 + q_2 q'_0 + q_3 q'_1 - q_1 q'_3)j + (q_0 q'_3 + q_3 q'_0 + q_1 q'_2 - q_2 q'_1)k \end{aligned}$$

It can be seen that multiplication is not commutative. The premultiplication or post-multiplication can be expressed as a linear operation.

$$qq' = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} q'0 \\ q'_1 \\ q'_2 \\ q'_3 \end{bmatrix} = \begin{bmatrix} q'_0 & -q'_1 & -q'_2 & -q'_3 \\ q'_1 & q'_0 & +q'_3 & -q'_2 \\ q'_2 & -q'_3 & q'_0 & q'_1 \\ q'_3 & q'_2 & -q'_1 & q'_0 \end{bmatrix} \begin{bmatrix} q0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{A.15}$$

A *conjugate* of a quaternion (q_0, q_1, q_2, q_3) is defined as $(q_0, -q_1, -q_2, -q_3)$. In other words, conjugate of

$$q = (q_0, \bar{q})$$

is given by

$$q^* = (q_0, -\bar{q})$$

. It follows that

$$qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

This non-negative number is called the 'Norm' of the quaternion q, denoted here by $N(q)$. Evidently, $N(q) = N(q^*)$.
Quaternionms with $N(q) = 1$ are called *unit quaternions.*

A.4.1 Some important results for quaternions

1. Multiplication is associative . $q(q'q'') = (qq')q'' = qq'q''$
2. $(qq')^* = q'^* \cdot q^*$
3. $N(qq') = N(q) \cdot N(q')$
4. $q^{-1} = q^*/N(q)$
5. $qq' = q_0q'_0 - \bar{q} \cdot \bar{q}' + q_0\bar{q}' + q'_0\bar{q}\bar{q} \times \bar{q}'$

Special cases of results 4. and 5. can be written for vector quaternions, as
4*a*. $q^{-1} = q^*$
5*a*. $qq' = -\bar{q} \cdot \bar{q}' + \bar{q} \times \bar{q}'$

A.4.2 Quaternion multiplications and rotations

Let (x_1, x_2, x_3) be the position vector of a point on the body in inertial frame. . For kinematical analysis, this can be generalized as a vector quaternion X given by $X = (0, x_1, x_2, x_3)$. With this, we have the following theorem.

Theorem 4 *For a vector quaternion $X = (0, \bar{x})$ and a unit quaternion q, the operation QX on X defined by $QX = qXq^*$ represents a rotation of $\bar{x}$ to another vector $\bar{x}'$ in 3-Dim space so that $QX = (0, \bar{x}')$*

Proof: Since X is a vector quaternion, we have $X + X^* = 0$ Again, denoting the vectorQX as X', we have $X' = qXq^*$.
This implies, from the result 2 in subsection A.4.1,

$$X'^* = qX^*q^*$$

Thus, adding these two, we get

$$X' + X'^* = q\,(X + X^*)\,q^* = 0$$

Hence, it is clear that X' is also a vector quaternion, equivalent to a 3-dimentional vector, expressible as

$$X' = QX = (0, \bar{x}')$$

Further, noting that q is unit-quaternion and using result 3, we have

$$N(X') = N(qXq^*) = N(q) \cdot N(X) \cdot N(q^*) = N(X)$$

which implies $x'^T x' = x^T x$ Hence the above transformation preserves the length of the position vector, implying it is a rotation. In fact , by actual multiplication, it can be verified that x' and x are related by the relation $x' = A(q)x$ as in Eq A.14. Further, if there are two successive rotations, from x to x', and from there to x'', represented as quaternion operations on X and X' by Q and Q' so that

$$X' = QX \quad ; \quad X'' = Q'X' = Q'QX$$

then we have $\quad X' = QX = qXq^*$
and $\quad X'' = Q'X' = q'X'q'^* = q'qXq^*q'^* = q''Xq''^*$
where $\quad q'' = q'q$.
hence, the result $Q'' \equiv QQ'$ of two successive rotations is represented by the quaternion $q'' = qq'$ which is the product of the quaternions representing the two rotations. This is a very useful property, and quaternion multiplication being purely algebraic operations, makes numerical processing of rigid-body kinematics much more easier and faster.